Logical Truth, Contradictions, Inconsistency, and Logical Equivalence

Logical Truth

- The semantical concepts of logical truth, contradiction, inconsistency, and logical equivalence in Predicate Logic are straightforward adaptations of the corresponding concepts in Sentence Logic.
- A closed sentence **X** of Predicate Logic is **logically true**, ⊨ **X**, if and only if **X** is true in all interpretations.
- The logical truth of a sentence is proved directly using general reasoning in semantics.
- Given soundness, one can also prove the logical truth of a sentence **X** by providing a derivation with no premises.
- The result of the derivation is that **X** is **theorem**, ⊢ **X**.

An Example

- ⊨ (∀x)Fx ⊃ (∃x)Fx.
  - Suppose d satisfies ‘(∀x)Fx’.
  - Then all x-variants of d satisfy ‘Fx’.
  - Since the domain D is non-empty, some x-variant of d satisfies ‘Fx’.
  - So d satisfies ‘(∃x)Fx’
  - Therefore d satisfies ‘(∀x)Fx ⊃ (∃x)Fx’, QED.
- ⊢ (∀x)Fx ⊃ (∃x)Fx.

1	(∀x)Fx	P
2	Fa	1 ∀ E
3	(∃x)Fx	1 ∃ I

Contradictions

- A closed sentence **X** of Predicate Logic is a **contradiction** if and only if **X** is false in all interpretations.
- A sentence **X** is false in all interpretations if and only if its negation ∼**X** is true on all interpretations.
- Therefore, one may directly demonstrate that a sentence is a contradiction by proving that its negation is a logical truth.
- If the ∼**X** of a sentence is a logical truth, then given completeness, it is a theorem, and hence ∼**X** can be derived from no premises.

- If a sentence **X** is such that if it is true in any interpretation, both **Y** and ∼**Y** are true in that interpretation, then **X** cannot be true on any interpretation.
- Given soundness, it follows that if **Y** and ∼**Y** are derivable from **X**, then **X** is a contradiction.

An Example

- '(∀x)(Fx & ∼Fx)' is a contradiction.
  - Suppose that a variable assignment **d** satisfies '(∀x)(Fx & ∼Fx)'.
  - Then all x-variants **d**[**u**/x] of **d** satisfy 'Fx & ∼Fx'.
  - Then **d**[**u**/x] satisfies 'Fx'.
  - Then **d**[**u**/x] satisfies '∼Fx'.
  - Then **d**[**u**/x] does not satisfy 'Fx', a contradiction.
  - Therefore, no variable assigment **d** satisfies '(∀x)(Fx & ∼Fx)', QED.

1	$(\forall x)(Fx \& \sim Fx)$	P
2	$Fa \& \sim Fa$	1 ∀ E
3	Fa	2 & E
4	$\sim Fa$	2 & E

Inconsistent Sets of Sentences

- A set of closed sentences of Predicate Logic is **consistent** if and only if there is an interpretation (a **model**) which makes all the sentences in the set true.
- A set of closed sentences of Predicate Logic is **inconsistent** just in case it is not consistent.
- Therefore, a set of closed sentences of Predicate Logic is inconsistent just in case it has no models.
- It follows from these definitions and that of a contradiction that a finite collection of sentences is inconsistent if and only if the conjunction of the sentences is a contradiction.
  - There is no model for a set of sentences ***X*** if and only if in every interpretation, each of the sentences of ***X*** is false.
  - This holds if and only if in every interpretation, the conjunction of the sentences of ***X*** is false.
  - This holds if and only if the conjunction of the sentences of ***X*** is a contradiction, QED.

Demonstrating Inconsistency

- The consistency of a set of closed sentences can be demonstrated by providing a single interpretation which makes all the sentences in the set true.
- A direct demonstration of inconsistency requires general reasoning.
- Inconsistency can be proved indirectly be either of two ways.
  - Derive a contradiction from the set of inconsistent sentences taken as premises.
  - Derive the negation of the conjunction of the sentences from no premises.

An Example

- {'(∀x)Fx', '∼(∃x)Fx'} is inconsistent.
  - Suppose there is an interpretation which makes both '(∀x)Fx' and '∼(∃x)Fx' true.
  - Then for a given variable assignment **d**, **d** satisfies both '(∀x)Fx' and '∼(∃x)Fx'.
  - Therefore, all x-variants **d**[**u**/x] of **d** satisfy 'Fx'.
  - So **d** satisfies '(∃x)Fx'.
  - It also follows that **d** satisfies '∼(∃x)Fx', which yields a contradiction.
  - So, there is no interpretation which makes both '(∀x)Fx', '∼(∃x)Fx' true, QED.

The Example Continued

1	$(\forall x)Fx \& \sim (\exists x)Fx$	P
2	$(\forall x)Fx$	1 & E
3	Fa	2 ∀ E
4	$(\exists x)Fa$	3 ∃ I
5	$\sim (\exists x)Fx$	1 & E

Logical Equivalence

- Two closed sentences of Predicate Logic are **logically equivalent** if and only if they have the same truth value in all interpretations.
- The logical equivalence of **X** and **Y** holds as well when **X** is true in all interpretations where **Y** is true, and **Y** is true in all interpretations where **X** is true.
- Alternatively, two sentences **X** and **Y** are logically equivalent just in case their bicondition **X** ≡ **Y** is a logical truth.

- Logical equivalence is demonstrated directly through general reasoning.
- It is proved indirectly with two derivations, each having one of the sentences as a premise and the other as a conclusion.