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The Completeness of Classical Propositional and Predicate Logic

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Contents

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Part I

Introduction

Kurt Gödel's *completeness* theorem for the first-order predicate calculus (1929–30) is one of the deepest classical results in metalogic, perhaps of deeper foundational significance than Gödel's own *incompleteness* theorem for arithmetic (1931). The theorem establishes the extensional equivalence of two very different notions of consequence for first-order formulas, *validity* and *provability*, the first one of which involves an unbounded universal quantification over the class of possible interpretations, while the second one merely asserts the existence of certain finite sequences of formulas.

The purpose of these notes is to chart a direct and self-contained route to a proof of the completeness theorem for first-order logic. Since the heart of the combinatorial argument is already present in the proof of the propositional case, the propositional case is treated independently. Once the proof strategy for the propositional case is laid out, the further complications required to handle existential witnesses in the predicate case are introduced. Thus, the completeness proof takes up the first two parts of what follows. The third part is devoted to further results and applications, while the last part collects some problem sets that introduce further material and might be useful for classroom use. It should be mentioned that where proofs are routine they have been merely sketched or even omitted (the full details to be supplied at the chalkboard), but any unexpected steps are explicitly mentioned.

Part II

The propositional case

Preliminaries

The language $\mathscr{L}_0$ of classical propositional logic comprises as basic symbols countably many propositional variables $p_0, p_1, \ldots$ as well as symbols for the connectives $\sim$ (not), $\supset$ (if ... then) and the two parentheses (and). We assume that these symbols are all distinct and no one occurs as a part of another one. We refer to the set of the propositional variables as the set At_0 of the atomic sentences.

1. Definition: The set F_0 of the formulas of the language $\mathscr{L}_0$ is inductively defined as the smallest set of strings over the alphabet containing At_0 and such that if φ and ψ are in F_0, then so are:

- ► $(\sim\varphi)$;
- ► $(\varphi \supset \psi)$.

2. Theorem: *Principle of induction on formulas*: If some property P holds of all the propositional variables and is such that it holds for $(\sim\varphi)$ and $(\varphi \supset \psi)$ whenever it holds for φ and ψ, then P holds of all formulas in F_0 .

Proof. Let S be the collection of all formulas with property P, so that, in particular, $S \subseteq F_0$. Then S contains the propositional variables and is closed under the connectives; since F_0 is the smallest such class, also $F_0 \subseteq S$. So $F_0 = S$, and every formula has propery P. □

3. Exercise: Prove that any formula in F_0 is *balanced*, in that it has as many left parentheses as right ones. Also, prove that no formula begins with $\sim$ and that no proper initial segment of a formula is a formula.

The formulas $(\varphi \vee \psi)$ and $(\varphi \wedge \psi)$ abbreviate $((\sim \varphi) \supset \psi)$ and $\sim(\varphi \supset (\sim \psi))$, respectively. Similarly, $\varphi \equiv \psi$ abbreviates $(\varphi \supset \psi) \wedge (\psi \supset \varphi)$. Parentheses around $\sim \varphi$ are usually dropped, with the understanding that $\sim$ binds the shortest formula that follows it; outermost parentheses are likewise usually dropped.

4. Theorem: *Unique Readability*: Any formula φ in F_0 has exactly one parsing as one of the following

1. p_n for some $p_n \in \mathsf{At}_0$
2. $(\sim \psi)$ for some ψ in F_0;
3. $(\psi \supset \theta)$ for some ψ, θ in F_0.

Moreover, such parsing is *unique*, in that, e.g., φ cannot have the form $(\sim \psi)$ *in two different ways*.

Proof. By induction on φ. For instance, suppose that φ has two distinct readings as $(\psi \supset \theta)$ and $(\psi' \supset \theta')$. If we remove the initial left parenthesis from each, then we see that one of them must be an initial segment of the other (or else one would be a *proper* initial segment of the other), and therefore ψ and ψ' must be the same sequence of symbols and hence by inductive hypothesis also the same formula. It also follows that θ and θ' are the same string, and therefore (by inductive hypothesis again) the same formula. □

5. Theorem: *Principle of definition by recursion*: for any set V and functions $\mathbf{i} : \mathsf{At}_0 \to V$ and h_1, h_2 from V and $V \times V$, respectively, into V, there exists exactly one function $f : F_0 \to V$ satisfying the following equations:

$$f(p_n) = \mathbf{i}(p_n)$$
$$f(\sim \varphi) = h_1(f(\varphi))$$
$$f(\varphi \supset \psi) = h_2(f(\varphi), f(\psi))$$

Proof. Let $\mathscr{F}$ be the class of all functions g such that:

- ► dom(g) is finite and closed under subformulas;
- ► whenever φ is in dom(g), g satisfies the equation corresponding to φ.

Put $f = \bigcup \mathscr{F}$. It is easy to see that:

(*a*) No two functions g and g' in $\mathscr{F}$ disagree on any of the arguments on which they are both defined. Hence, f is well-defined as a function. (This requires Unique Readability.)

(*b*) f satisfies the equations.

(c) f is unique.

(d) f is total, i.e., $\mathrm{dom}(f) = F_0$.

These are established by induction on formulas. □

6. Definition: *Uniform substitution*. If φ and ψ are formulas, and p_i is a propositional variable, then $\varphi[\psi/p_i]$ denotes the result of replacing each occurrence of p_i by an occurrence of ψ in φ; similarly, the simultaneous substitution of $p_1, \dots, p_n$ by formulas $\psi_1, \dots, \psi_n$ is denoted by $\varphi[\psi_1/p_1, \dots, \psi_n/p_n]$.

7. Exercise: Give a mathematically rigorous definition of $\varphi[\psi/p_i]$ using Theorem 5.

Semantics for propositional logic

8. Definition: Let $\{\mathsf{t}, \mathsf{f}\}$ be the set of the two truth values, "true" and "false." A *valuation* for $\mathscr{L}_0$ is a function v assigning either t or f to the propositional variables of the language, i.e., $v : \mathsf{At}_0 \to \{\mathsf{t}, \mathsf{f}\}$.

9. Theorem: Every valuation v can be "lifted" to a unique function $\overline{v} : F_0 \to \{\mathsf{t}, \mathsf{f}\}$ such that:

$$\overline{v}(p_n) = v(p_n);$$

$$\overline{v}({\sim}\varphi) = \begin{cases} \mathsf{t} & \text{if } \overline{v}(\varphi) = \mathsf{f}; \\ \mathsf{f} & \text{otherwise}; \end{cases}$$

$$\overline{v}(\varphi \supset \psi) = \begin{cases} \mathsf{t} & \text{if } \overline{v}(\varphi) = \mathsf{f} \text{ or } \overline{v}(\psi) = \mathsf{t}; \\ \mathsf{f} & \text{if } \overline{v}(\varphi) = \mathsf{t} \text{ and } \overline{v}(\psi) = \mathsf{f}. \end{cases}$$

Proof. Apply Theorem 5. □

10. Exercise: Show that the valuation clauses for $\supset$ and $\sim$ give the right truth tables for $\wedge$ and $\vee$:

φ	ψ	$\varphi \wedge \psi$	$\varphi \vee \psi$
t	t	t	t
t	f	f	t
f	t	f	t
f	f	f	f

11. Theorem: *Local Determination*: Suppose that v_1 and v_2 are valuations that agree on the propositional letters occurring in φ, i.e., $v_1(p_n) = v_2(p_n)$ whenever p_n occurs in φ. Then they also agree on φ, i.e., $\overline{v}_1(\varphi) = \overline{v}_2(\varphi)$.

Proof. By induction on φ. □

12. Definition: The following are semantic notions:

- A formula φ is *satisfiable* if for some v, $\overline{v}(\varphi) = \mathsf{t}$; it is *unsatisfiable* if for no v it holds $\overline{v}(\varphi) = \mathsf{t}$;
- A formula φ is a *tautology* if $\overline{v}(\varphi) = \mathsf{t}$ for all valuations v;

- ► A formula φ is *contingent* if it is satisfiable but not a tautology;
- ► if Γ is a set of formulas, $\Gamma \models \varphi$ ("Γ entails φ") if and only if $\overline{v}(\varphi) = \mathsf{t}$ for every valuation v on which $\overline{v}(\psi) = \mathsf{t}$ for every $\psi \in \Gamma$.
- ► if Γ is a set of formulas, Γ is *satisfiable* if there is a valuation v on which $\overline{v}(\psi) = \mathsf{t}$ for every $\psi \in \Gamma$, and Γ is unsatisfiable otherwise.

13. Exercise: The following can all be proved with little more than "definition chasing":

(*a*) φ is a tautology if and only if $\varnothing \models \varphi$;

(*b*) $\Gamma \models \varphi$ if and only if $\Gamma \cup \{\sim \varphi\}$ is unsatisfiable;

(*c*) if $\Gamma \models \varphi$ and $\Gamma \models \varphi \supset \psi$ then $\Gamma \models \psi$;

(*d*) if Γ is satisfiable then every finite subset of Γ is also satisfiable;

(*e*) *Monotony*: if $\Gamma \subseteq \Delta$ and $\Gamma \models \varphi$ then also $\Delta \models \varphi$;

(*f*) *Cut*: if $\Gamma \models \varphi$ and $\Delta \cup \{\varphi\} \models \psi$ then $\Gamma \cup \Delta \models \psi$;

(*g*) *Deduction Theorem*: $\Gamma \models \varphi \supset \psi$ if and only if $\Gamma \cup \{\varphi\} \models \psi$.

Proof theory for propositional logic

14. Definition: The set **Ax** of the *axioms* of propositional logic comprises all formulas of the following forms:

[Ax1] $\varphi \supset (\psi \supset \varphi)$;

[Ax2] $(\varphi \supset (\psi \supset \chi)) \supset ((\varphi \supset \psi) \supset (\varphi \supset \chi))$;

[Ax3] $(\sim \varphi \supset \sim \psi) \supset (\psi \supset \varphi)$.

15. Definition: If Γ is a set of formulas of $\mathscr{L}_0$ and φ a formula, then a *proof* of φ from Γ is a finite sequence $\varphi_1, \ldots, \varphi_n$ of formulas such that $\varphi_n = \varphi$ and for each $i \leq n$ one of the following holds:

- ► φ_i is an axiom; or
- ► $\varphi_i \in \Gamma$; or
- ► there are $j, k < i$ such that φ_j is $\varphi_k \supset \varphi_i$.

The last clause just says that φ_i can be obtained from previously occurring φ_k and $\varphi_k \supset \varphi_i$ by *Modus Ponens* (MP). We write $\Gamma \vdash \varphi$ ("Γ proves φ," or "φ is provable from Γ") to mean that there is a proof of φ from Γ. When Γ is empty, we write $\vdash \varphi$ to mean $\varnothing \vdash \varphi$. When all the formulas in Γ are specified, curly brackets may be dropped. The *theorems* of Γ consist of all the formulas provable from Γ: $\mathsf{Thm}_\Gamma = \{\varphi : \Gamma \vdash \varphi\}$.

16. Proposition: The following are provable:

(*a*) *Transitivity*: $\{\varphi \supset \psi, \psi \supset \theta\} \vdash \varphi \supset \theta$;

(*b*) *Identity*: $\vdash \varphi \supset \varphi$.

Proof. For part (a), the following instances of **Ax1** and **Ax2** are needed:

$$(\psi \supset \theta) \supset (\varphi \supset (\psi \supset \theta));$$
$$(\varphi \supset (\psi \supset \theta)) \supset ((\varphi \supset \psi) \supset (\varphi \supset \theta)).f$$

For part (b), consider the instance of **Ax1**: $\varphi \supset ((\varphi \supset \varphi) \supset \varphi)$ and distribute the outermost implication by **Ax2**. □

17. Proposition: For any set Γ of formulas:

(a) if φ is an axiom then $\Gamma \vdash \varphi$;
(b) *Reflexivity*: if $\varphi \in \Gamma$ then $\Gamma \vdash \varphi$;
(c) *Closure under Modus Ponens*: if $\Gamma \vdash \varphi$ and $\Gamma \vdash \varphi \supset \psi$ then $\Gamma \vdash \psi$
(d) *Monotony*: if $\Gamma \subseteq \Delta$ and $\Gamma \vdash \varphi$ then also $\Delta \vdash \varphi$;
(e) $\Gamma \vdash \varphi$ if and only if there is a *finite* subset Γ_0 of Γ such that $\Gamma_0 \vdash \varphi$.

18. Theorem: Thm_Γ is the smallest set of formulas containing both the axioms and every formula in Γ, and that is closed under *Modus Ponens* (from φ and $\varphi \supset \psi$ infer ψ).

Proof. We know from Proposition 17 that Thm_Γ has the desired properties; so we need to show that it is the smallest such set. Let A be any other set of formulas containing the axioms and Γ and that is closed under *Modus Ponens*. Prove that $\mathsf{Thm}_\Gamma \subseteq A$ by induction on the length of a proof of φ from Γ. □

19. Corollary: *Principle of induction on theorems*: any property P that holds of the axioms, of formulas in Γ, and is preserved by *Modus Ponens* holds of every formula in $\mathsf{Thm}(\Gamma)$.

20. Theorem: *Deduction Theorem*: $\Gamma \cup \{\varphi\} \vdash \psi$ if and only if $\Gamma \vdash \varphi \supset \psi$.

Proof. The "if" direction is immediate; if $\Gamma \vdash \varphi \supset \psi$ then also $\Gamma \cup \{\varphi\} \vdash \varphi \supset \psi$ by Monotony, so there is a proof of $\varphi \supset \psi$ from $\Gamma \cup \{\varphi\}$, and one more application of *Modus Ponens* gives $\Gamma \cup \{\varphi\} \vdash \psi$.

For the converse, proceed by induction on theorems. If $\psi \in \Gamma$ or ψ is an axiom then also $\Gamma \vdash \psi \supset (\varphi \supset \psi)$ by **Ax1**, and *Modus Ponens* (and Monotony) gives $\Gamma \cup \{\varphi\} \vdash \varphi \supset \psi$; and if $\psi \in \{\varphi\}$ then $\Gamma \vdash \varphi \supset \psi$ because the last sentence is the same as $\varphi \supset \varphi$.

For the inductive step, suppose ψ is obtained by *modus ponens* from $\theta \supset \psi$ and θ. Then $\Gamma \cup \{\varphi\} \vdash \theta \supset \psi$ and $\Gamma \cup \{\varphi\} \vdash \theta$. By the inductive hypothesis, both

$$\Gamma \vdash \varphi \supset (\theta \supset \psi);$$
$$\Gamma \vdash \varphi \supset \theta.$$

But also

$$\Gamma \vdash (\varphi \supset (\theta \supset \psi)) \supset ((\varphi \supset \theta) \supset (\varphi \supset \psi)),$$

by **Ax2**, and two applications of *Modus Ponens* give $\Gamma \vdash \varphi \supset \psi$, as required. □

Notice how **Ax1** and **Ax2** were chosen precisely so that the Deduction Theorem would hold. The following proposition collects useful facts about provability that will be needed in the next section.

21. Proposition: The following are all provable:

(*a*) $\vdash (\varphi \supset \psi) \supset ((\psi \supset \theta) \supset (\varphi \supset \theta))$;

(*b*) *Contraposition*: if $\Gamma \cup \{\sim\varphi\} \vdash \sim\psi$ then $\Gamma \cup \{\psi\} \vdash \varphi$;

(*c*) *Ex Falso Quodlibet*: $\{\varphi, \sim\varphi\} \vdash \psi$;

(*d*) *Double Negation*: $\{\sim\sim\varphi\} \vdash \varphi$;

(*e*) if $\Gamma \vdash \sim\sim\varphi$ then $\Gamma \vdash \varphi$;

Proof. Part (*a*) follows from Prop. 16, part (*a*) by two applications of the Deduction Theorem. For part (*b*):

1.	$\Gamma \cup \{\sim\varphi\} \vdash \sim\psi$	hyp.
2.	$\Gamma \vdash \sim\varphi \supset \sim\psi$	Ded. Thm., 1
3.	$\Gamma \vdash (\sim\varphi \supset \sim\psi) \supset (\psi \supset \varphi)$	**Ax3** and Monotony
4.	$\Gamma \vdash \psi \supset \varphi$	MP, 2, 3
5.	$\Gamma \cup \{\psi\} \vdash \varphi$	Ded. Thm., 4

For part (*c*) we have $\{\sim\varphi, \sim\psi\} \vdash \sim\varphi$ by Reflexivity, and $\{\varphi, \sim\varphi\} \vdash \psi$ follows by Prop 17, part (*b*). Part (*d*): since $\{\sim\sim\varphi, \sim\varphi\} \vdash \sim\sim\sim\varphi$ by *Ex Falso Quodlibet*, apply (*b*). Now for part (*e*), $\Gamma \vdash \sim\sim\varphi \supset \varphi$ by the previous part, the Deduction Theorem, and Monotony, so if $\Gamma \vdash \sim\sim\varphi$ also $\Gamma \vdash \varphi$ by MP. □

22. Theorem: *Cut*: if $\Gamma \vdash \varphi$ and $\Delta, \varphi \vdash \theta$ then $\Gamma \cup \Delta \vdash \theta$.

Proof. The following shows that the conclusion is derivable:

1.	$\Gamma \vdash \varphi$	hyp.
2.	$\Gamma \cup \Delta \vdash \varphi$	Monotony, 1
3.	$\Delta, \varphi \vdash \theta$	hyp.
4.	$\Gamma \cup \Delta, \varphi \vdash \theta$	Monotony, 3
5.	$\Gamma \cup \Delta \vdash \varphi \supset \theta$	Ded. Thm., 4
6.	$\Gamma \cup \Delta \vdash \theta$	MP, 2, 5

□

23. Lemma: $\varphi \supset \psi, \varphi \vdash \psi$.

Proof. By Reflexivity, $\varphi \supset \psi \vdash \varphi \supset \psi$. Apply the Deduction Theorem. □

24. Lemma: $\varphi \supset \sim\varphi \vdash \sim\varphi$.

Proof. The following shows the conclusion is derivable.

1.	$\sim\sim\varphi \vdash \varphi$	Double Neg.
2.	$\sim\sim\sim\sim\varphi \vdash \sim\sim\varphi$	Double Neg.
3.	$\sim\sim(\varphi \supset \sim\varphi) \vdash \varphi \supset \sim\varphi$	Double Neg.
4.	$\sim\varphi \vdash \sim\sim\sim\varphi$	Contraposition, 2
5.	$\sim\sim(\varphi \supset \sim\varphi), \varphi \vdash \sim\varphi$	Ded. Thm., 3
6.	$\sim\sim\varphi, \sim\sim(\varphi \supset \sim\varphi) \vdash \sim\varphi$	Cut, 1, 5
7.	$\sim\sim\varphi, \sim\sim(\varphi \supset \sim\varphi) \vdash \sim\sim\sim\varphi$	Cut 4, 6
8.	$\sim\sim\varphi \vdash \sim(\varphi \supset \sim\varphi)$	Contraposition, 7
9.	$(\varphi \supset \sim\varphi) \vdash \sim\varphi$	Contraposition, 8

□

25. Proposition: The following hold:

(*a*) $\psi \vdash \sim\sim\psi$

(*b*) $\varphi \supset \sim\psi, \psi \vdash \sim\varphi$

(*c*) $\varphi \supset \psi, \varphi \supset \sim\psi \vdash \sim\varphi$.

(*d*) if $\Gamma \cup \{\varphi\} \vdash \psi$ and $\Gamma \cup \{\varphi\} \vdash \sim\psi$ then $\Gamma \vdash \sim\varphi$.

Proof. For part (*a*), we have $\sim\sim\sim\psi \vdash \sim\psi$ by double negation, whence $\psi \vdash \sim\sim\psi$ by Prop. 21, part (*b*). For part (*b*) we have:

1.	$\vdash (\sim\sim\varphi \supset \sim\sim\sim\psi) \supset (\sim\sim\psi \supset \sim\varphi)$	**Ax3**
2.	$\sim\sim\varphi \supset \sim\sim\sim\psi, \sim\sim\psi \vdash \sim\varphi$	Ded. Thm., twice, 1
3.	$\psi \vdash \sim\sim\psi$	Double Neg.
4.	$\sim\sim\varphi \supset \sim\sim\sim\psi, \psi \vdash \sim\varphi$	Cut, 2, 3
5.	$\sim\psi \supset \sim\sim\sim\psi, \varphi \supset \sim\psi \vdash \varphi \supset \sim\sim\sim\psi$	Transitivity
6.	$\vdash \sim\psi \supset \sim\sim\sim\psi$	Double Neg. + Ded. Thm.
7.	$\varphi \supset \sim\psi \vdash \varphi \supset \sim\sim\sim\psi$	5, 6, Cut
8.	$\vdash \sim\sim\varphi \supset \varphi$	Double Neg. + Ded. Thm.
9.	$\sim\sim\varphi \supset \varphi, \varphi \supset \sim\sim\sim\psi \vdash \sim\sim\varphi \supset \sim\sim\sim\psi$	Transitivity
10.	$\varphi \supset \sim\sim\sim\psi \vdash \sim\sim\varphi \supset \sim\sim\sim\psi$	Cut, 8, 9
11.	$\varphi \supset \sim\psi \vdash \sim\sim\varphi \supset \sim\sim\sim\psi$	Cut, 7 10
12.	$\varphi \supset \sim\psi, \psi \vdash \sim\varphi$	Cut, 4, 11

Part (*c*) follows because:

1.	$\varphi \supset \psi, \varphi \vdash \psi$	Lemma 23
2.	$\varphi \supset \sim\psi, \psi \vdash \sim\varphi$	part (*b*)
3.	$\varphi \supset \psi, \varphi \supset \sim\psi, \varphi \vdash \sim\varphi$	Cut, 1, 2
4.	$\varphi \supset \psi, \varphi \supset \sim\psi \vdash \varphi \supset \sim\varphi$	Ded. Thm., 3
5.	$\varphi \supset \sim\varphi \vdash \sim\varphi$	Lemma 24
6.	$\varphi \supset \psi, \varphi \supset \sim\psi \vdash \sim\varphi$	Cut, 4, 5.

Finally, for part (d): from the hypotheses by the Deduction Theorem, $\Gamma \vdash \varphi \supset \psi$ and $\Gamma \vdash \varphi \supset \sim\psi$; from part ($c$) and Monotony, $\Gamma, \varphi \supset \psi, \varphi \supset \sim\psi \vdash \sim\varphi$; two applications of Cut give the desired result. □

26. Definition: A set Γ of formulas is *consistent* if and only if there is a formula φ such that $\Gamma \nvdash \varphi$; it is *inconsistent* otherwise.

27. Proposition: Γ is inconsistent if and only if there is a formula φ such that both $\Gamma \vdash \varphi$ and $\Gamma \vdash \sim\varphi$.

Proof. The "only if" direction is obvious. For the converse, suppose that $\Gamma \vdash \varphi$ and $\Gamma \vdash \sim\varphi$. Then by Proposition 21, part (c) and Monotony, $\Gamma \cup \{\varphi, \sim\varphi\} \vdash \psi$ for any formula ψ, and now two applications of Cut give $\Gamma \vdash \psi$ for any ψ, so Γ is inconsistent. □

28. Proposition: $\Gamma \vdash \varphi$ if and only if $\Gamma \cup \{\sim\varphi\}$ is inconsistent.

Proof. If $\Gamma \vdash \varphi$ then also $\Gamma, \sim\varphi \vdash \varphi$ by Monotony, and $\Gamma, \sim\varphi \vdash \sim\varphi$ by Reflexivity, so $\Gamma \cup \{\sim\varphi\}$ is inconsistent. Conversely, suppose $\Gamma \cup \{\sim\varphi\}$ is inconsistent. Then by Proposition 27, $\Gamma, \sim\varphi \vdash \theta$ and $\Gamma, \sim\varphi \vdash \sim\theta$ for some θ. By Proposition 25 part (d), $\Gamma \vdash \sim\sim\varphi$. But also $\sim\sim\varphi \vdash \varphi$, by Double Negation, so that by Cut, $\Gamma \vdash \varphi$. □

29. Proposition: If Γ is consistent, then for any formula φ, either $\Gamma \cup \{\varphi\}$ is consistent or $\Gamma \cup \{\sim\varphi\}$ is consistent.

Proof. From Proposition 28 we have that if $\Gamma \cup \{\sim\varphi\}$ is inconsistent, then $\Gamma \vdash \varphi$; if $\Gamma \cup \{\varphi\}$ is also inconsistent, then $\Gamma \cup \{\varphi\} \vdash \psi$ for any ψ. But then by Cut, $\Gamma \vdash \psi$ for any ψ, so Γ is inconsistent. □

30. Proposition: Γ is consistent if and only if every finite subset $\Gamma_0 \subseteq \Gamma$ is consistent.

Proof. For the non-trivial direction: if Γ is inconsistent, then $\Gamma \vdash \psi$ and $\Gamma \vdash \sim\psi$ for some ψ; each proof involves only finitely many formulas from Γ. Collect the ones occurring in the first proof into the finite set Γ_1, and those occurring in the second proof into the finite set Γ_2. Then $\Gamma_0 = \Gamma_1 \cup \Gamma_2$ is a finite subset of Γ that is inconsistent. □

31. Exercise: Show that the following hold by *exhibiting* proofs from the axioms (i.e., without using meta-theoretic facts such as Cut, Monotony, the Deduction Theorem, etc.):

(a) $\{\varphi \supset \psi, \psi \supset \theta\} \vdash \varphi \supset \theta$;
(b) $\vdash \varphi \supset \varphi$;
(c) $\vdash \sim\varphi \supset (\varphi \supset \psi)$;
(d) $\vdash \varphi \supset (\sim\varphi \supset \psi)$.

Soundness and Completeness of propositional logic

32. Theorem: *Soundness*: If $\Gamma \vdash \varphi$ then $\Gamma \models \varphi$.

Proof. By induction on theorems. If φ is an axiom then $\models \varphi$ and hence vacuously $\Gamma \models \varphi$. Similarly if $\varphi \in \Gamma$, $\Gamma \models \varphi$. For the inductive step, suppose φ is obtained by *Modus Ponens* from θ and $\theta \supset \varphi$; also assume as inductive hypothesis that the theorem holds for $\theta \supset \varphi$ and θ. Exercise 13, part (c) gives the desired result. □

33. Corollary: If Γ is satisfiable, then Γ is consistent. Hence, propositional logic is consistent.

34. Definition: A set Γ of formulas is *maximally consistent* if it is consistent and if Δ is a consistent set such that $\Gamma \subseteq \Delta$ then $\Gamma = \Delta$.

35. Proposition: *Truth Lemma*: let Γ be maximally consistent; then:

(a) $\Gamma \vdash \varphi$ if and only if $\varphi \in \Gamma$;

(b) $\varphi \in \Gamma$ if and only if $\sim\varphi \notin \Gamma$;

(c) $\varphi \supset \psi \in \Gamma$ if and only if either $\varphi \notin \Gamma$ or $\psi \in \Gamma$.

Proof. By the way of example, let $\varphi \in \Gamma$; if also $\sim\varphi \in \Gamma$, then Γ is inconsistent; and if neither φ nor $\sim\varphi$ is in Γ then by Proposition 29 one of $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \{\sim\varphi\}$ is consistent, which means that Γ is not maximal. □

The left-to-right of item (a) of the Truth Lemma is the *deductive closure* of Γ, i.e., if $\Gamma \vdash \varphi$ then $\varphi \in \Gamma$.

36. Theorem: If Γ is consistent then Γ is satisfiable.

Proof. Let $\varphi_0, \varphi_1, \ldots$ be an exhaustive listing of all the formulas of the language. Recursively define an increasing sequence of sets of formulas $\Gamma_0, \Gamma_1, \ldots$, by putting:

$$\Gamma_0 = \Gamma$$

$$\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{\varphi_n\} & \text{if } \Gamma_n \cup \{\varphi_n\} \text{ is consistent;} \\ \Gamma_n \cup \{\sim\varphi_n\} & \text{otherwise.} \end{cases}$$

Then define:

$$\Gamma^* = \bigcup_{0 \leq n} \Gamma_n.$$

The proof now proceeds by establishing, in turn, the following facts:

(a) For each n, the set Γ_n is consistent (by induction on n, using Proposition 29);

(b) Γ^* is consistent;

(c) Γ^* is maximal.

Then define a valuation v by putting $v(p_i) = \mathsf{t}$ if and only if $p_i \in \Gamma^*$. By induction on φ it is then shown that membership in Γ^* coincides with truth according to v also for more complex sentences: $\overline{v}(\varphi) = \mathsf{t}$ if and only if $\varphi \in \Gamma^*$. In particular, v satisfies Γ^*, and since $\Gamma \subseteq \Gamma^*$, also Γ is satisfiable, as desired. □

37. Corollary: *Completeness*: If $\Gamma \models \varphi$ then $\Gamma \vdash \varphi$.

Proof. If $\Gamma \nvdash \varphi$ then $\Gamma \cup \{\sim \varphi\}$ is consistent, by Proposition 28. So by the theorem $\Gamma \cup \{\sim \varphi\}$ is satisfiable, so $\Gamma \not\models \varphi$. □

38. Proposition: *Compactness Theorem*: Γ is satisfiable if and only if every *finite* subset Γ_0 of Γ is satisfiable.

Proof. Γ is unsatisfiable if and only if it is inconsistent, if and only if some finite subset Γ_0 of Γ is inconsistent, if and only if some finite subset Γ_0 of Γ is unsatisfiable. □

39. Corollary: $\Gamma \models \varphi$ if and only if for some finite subset Γ_0 of Γ, $\Gamma_0 \models \varphi$.

Part III

The predicate case

Preliminaries

A language $\mathscr{L}_1$ of classical predicate logic comprises the connectives $\sim$ and $\supset$, the universal quantifier $\forall$, parentheses (and) as well as:

- denumerably many individual variables $v_0, v_1, \ldots$;
- countably many (i.e., finitely or denumerably many) individual constants $c_0, c_1, \ldots$;
- for each $n > 0$, countably many n-place predicate symbols, including at least the 2-place symbol $\doteq$ for identity;
- for each $n > 0$, countably many n-place function symbols.

40. Definition: We define the sets comprising the terms, atomic formulas and formulas of $\mathscr{L}_1$:

- The set T_1 of the terms of $\mathscr{L}_1$ is defined as the smallest set containing the constants, the variables, and such that if $t_1, \ldots, t_n$ are terms and f is an n-place function symbol, then $f t_1 \ldots t_n$ is also a term.
- The set At_1 of the atomic formulas of $\mathscr{L}_1$ comprises all expressions of the form $P t_1 \ldots t_n$, where $t_1, \ldots, t_n$ are terms and P is an n-place predicate symbol, as well as all expressions of the form $t_1 \doteq t_2$.
- The set F_1 of the formulas of $\mathscr{L}_1$ is defined as the smallest set containing the atomic formulas and such that if φ and ψ are formulas and x is a variable, then $(\sim \varphi)$, $(\varphi \supset \psi)$ and $(\forall x \varphi)$ are formulas (the formula φ is the *scope* of the quantifier $\forall x$).

We adopt the same conventions for dropping parentheses as in the propositional case as well as the same abbreviations for $(\varphi \wedge \psi)$ and $(\varphi \vee \psi)$. Moreover, we abbreviate $\sim\forall x \sim\varphi$ by $\exists x\varphi$. Just as in the propositional case, we have a *principle of induction* on both terms and formulas, as well as a *principle of definition by recursion* (also on both terms and formulas).

41. Definition: The following notions relate to the occurrence of variables in formulas:

1. A variable x occurs *free* in a formula φ if it does not fall within the scope of a quantifier $\forall x$. (If x falls within the scope of $\forall x$, then x is *bound* by $\forall x$.) This can be defined recursively on the complexity of φ: x is always free in φ if $\varphi \in \mathsf{At}_1$; x is free in $\sim\psi$ or $\psi \supset \theta$ if it is free in ψ or θ; and x is free in $\forall y\psi$ if it is free in ψ and not the same variable as y.
2. If φ is a formula and $x_1, \ldots, x_n$ are distinct variables, we denote by $\varphi(x_1, \ldots, x_n)$ the fact that all of the variables occurring free in φ are among $x_1, \ldots, x_n$.
3. If no variable occurs free in φ, then φ is a *sentence*.

42. Definition: We define a *substitution instance for x in t*, $t[t'/x]$, as the result of replacing t' for every occurrence of x in t, recursively on the complexity of t: if t is a constant c or a variable other than x then $t[t'/x]$ is just t; if t is x then $t[t'/x]$ is t'; and if t is $f\,t_1 \ldots t_n$ then $t[t'/x]$ is $f\,t_1[t'/x] \ldots t_n[t'/x]$.

43. Definition: We define a *substitution instance for x* in φ, $\varphi[t/x]$, as the result of replacing t for every free occurrence of x in φ, recursively on the complexity of φ:

- if φ is $P t_1 \ldots t_n$ or $t_1 \doteq t_2$ then $\varphi[t/x]$ is $P t_1[t/x] \ldots t_n[t/x]$ or $t_1[t/x] \doteq t_2[t/x]$, respectively;
- if φ is $\sim\psi$ or $\psi \supset \theta$ then $\varphi[t/x]$ is $\sim\psi[t/x]$ or $\psi[t/x] \supset \theta[t/x]$, respectively;
- if φ is $\forall y\psi$ (where y is a variable other than x), then $\varphi[t/x]$ is $\forall y\psi[t/x]$;
- if φ is $\forall x\psi$ then $\varphi[t/x]$ is just φ.

More generally, we understand Definition 43 to extend to substitution for a constant c.

44. Definition: A term t is *free for x in φ* if x does not occur in φ within the scope of a quantifier $\forall y$ binding a variable y occurring in t.

Needless to say, the previous definition can be more precisely given as a recursion on φ.

Semantics of predicate logic

45. Definition: A *structure* $\mathfrak{A}$ for $\mathscr{L}_1$ provides a non empty *domain* or *universe* $|\mathfrak{A}| = A$ as well as:

(*a*) for each constant symbol c of $\mathscr{L}_1$, an element $c^{\mathfrak{A}} \in A$;
(*b*) for each n-place predicate symbol P of $\mathscr{L}_1$ (other than $\doteq$), an n-ary relation $P^{\mathfrak{A}} \subseteq A^n$;
(*c*) for each n-place function symbol f of $\mathscr{L}_1$, an n-ary function $f^{\mathfrak{A}} : A^n \to A$.

46. Definition: An *assignment* to the variables is a function s that assigns to each variable x a member $s(x)$ of A. If s is an assignment, x a variable, and $a \in A$, then $s(a/x)$ is the assignment defined as follows:

$$s(a/x)(y) = \begin{cases} a, & \text{if } y \text{ is } x; \\ s(y), & \text{otherwise.} \end{cases}$$

The function $s(a/x)$ is the result of shifting s at x and is called an x-variant of s.

47. Proposition: Each assignment can be "lifted" to a function $\bar{s} : T_1 \to A$ assigning a member $\bar{s}(t)$ of the domain A to each term t in T_1, recursively as follows:

$$\begin{aligned} \bar{s}(x) &= s(x), \\ \bar{s}(c) &= c^{\mathfrak{A}}, \\ \bar{s}(f\,t_1,\ldots,t_n) &= f^{\mathfrak{A}}(\bar{s}(t_1),\ldots,\bar{s}(t_n)). \end{aligned}$$

48. Theorem: *Local Determination I*: If s_1 and s_2 are assignments that agree on the variables occurring in a term t, then they assign the same denotation to t, i.e., $\overline{s_1}(t) = \overline{s_2}(t)$.

Proof. By induction on t. □

49. Definition: The notion of *satisfaction* of a formula $\varphi(x_1,\ldots,x_n)$ by an assignment s in a structure $\mathfrak{A}$, written $\mathfrak{A} \models \varphi[s]$, is defined by recursion on φ:

- $\mathfrak{A} \models P t_1 \ldots t_n[s]$ if and only if $\langle \bar{s}(t_1),\ldots,\bar{s}(t_n)\rangle \in P^{\mathfrak{A}}$;
- $\mathfrak{A} \models (t_1 \doteq t_2)[s]$ if and only if $\bar{s}(t_1) = \bar{s}(t_2)$;
- $\mathfrak{A} \models (\sim \varphi)[s]$ if and only if $\mathfrak{A} \not\models \varphi[s]$;
- $\mathfrak{A} \models (\varphi \supset \psi)[s]$ if and only if either $\mathfrak{A} \not\models \varphi[s]$ or $\mathfrak{A} \models \psi[s]$;
- $\mathfrak{A} \models (\forall x \varphi)[s]$ if and only if $\mathfrak{A} \models \varphi[s(a/x)]$ for every $a \in A$.

50. Theorem: *Local Determination II*: If s_1 and s_2 are assignments that agree on the variables occurring *free* in a formula φ, then $\mathfrak{A} \models \varphi[s_1]$ if and only if $\mathfrak{A} \models \varphi[s_2]$.

Proof. By induction on φ. □

51. Corollary: If φ is a sentence, then $\mathfrak{A} \models \varphi[s]$ for some s if and only if $\mathfrak{A} \models \varphi[s]$ for every s.

52. Corollary: $\mathfrak{A} \models \varphi[s]$ for every structure $\mathfrak{A}$ and assignment s, if and only if $\mathfrak{A} \models \forall x \varphi[s]$ for every structure $\mathfrak{A}$ and assignment s.

53. Definition: The following notions concern the satisfaction of formulas and sets thereof:

1. A sentence φ is *true* in a structure $\mathfrak{A}$, written $\mathfrak{A} \models \varphi$, if and only if for some (equivalently: every) assignment s, $\mathfrak{A} \models \varphi[s]$.
2. Where Γ is a set of formulas, s *satisfies* Γ in $\mathfrak{A}$, written $\mathfrak{A} \models \Gamma[s]$, if and only if $\mathfrak{A} \models \varphi[s]$ for every φ in Γ.
3. Γ *entails* φ, written $\Gamma \models \varphi$, if and only if for every structure $\mathfrak{A}$ and assignment s, if $\mathfrak{A} \models \Gamma[s]$, then $\mathfrak{A} \models \varphi[s]$.

54. Substitution Theorem: Let $\mathfrak{A}$ be a structure:

1. for every assignment s in $\mathfrak{A}$ and for all termsfor z in φ; x, y not in t, t' or φ t and t', $\bar{s}(t[t'/x]) = \overline{s(\bar{s}(t')/x)}(t)$;

2. if t is free for x in φ, then for every assignment s: $\mathfrak{A} \models \varphi[t/x][s]$ if and only if $\mathfrak{A} \models \varphi[s(\bar{s}(t)/x)]$

Proof. By induction on t and φ. At the inductive step for $\forall y\varphi$ (with y a variable other than x), the inductive hypothesis is employed as follows: since t is free for x in $\forall y\varphi$, the variable y does not occur in t, and hence s and $s(a/y)$ agree on t, by Local Determination. Notice that the full inductive hypothesis quantifying over all s is needed, in order to instantiate with $s(a/y)$. □

Proof theory of predicate logic

55. Definition: A formula φ of predicate logic is a *tautological instance* if and only if there is a tautology $\psi(p_1,\dots,p_n)$ of propositional logic and formulas $\theta_1,\dots,\theta_n$ of predicate logic such that $\varphi = \psi[\theta_1/p_1,\dots,\theta_n/p_n]$, i.e., φ is the result of substituting θ_i for each p_i in $\psi(p_1,\dots,p_n)$.

56. Definition: The set Ax of the *axioms* of predicate logic comprises all formulas obtained by prefixing any number of (or no) universal quantifiers to the following:

[Ax0] φ, where φ is a tautological instance;

[Ax1] $\forall x\psi \supset \psi[t/x]$, if t is free for x in ψ;

[Ax2] $\forall x(\varphi \supset \psi) \supset (\forall x\varphi \supset \forall x\psi)$;

[Ax3] $\psi \supset \forall x\psi$, if x is *not* free in ψ;

[Ax4] $x \doteq x$;

[Ax5] $x \doteq y \supset (\psi[x/z] \supset \psi[y/z])$, if both x and y are free for z in ψ.

57. Definition: A *proof* from Γ is a finite sequence of formulas, each one of which is either an axiom, or a member of Γ, or is obtained by previous formulas by *Modus Ponens*. A formula φ is *provable* from Γ, written $\Gamma \vdash \varphi$, if there is a proof from Γ ending in φ.

58. Definition: $\mathsf{Thm}_\Gamma = \{\varphi : \Gamma \vdash \varphi\}$.

59. Proposition: Thm_Γ is the smallest set of formulas containing Γ and the axioms, and is closed under *Modus Ponens*. Accordingly, we have a principle of proof by *induction on the theorems of* Γ.

For example, let us show that $\forall x\psi \vdash \exists x\psi$ (where the formula on the right of the turnstile is just an abbreviation for $\sim\forall x\sim\psi$):

1.	$\forall x\sim\psi \supset \sim\psi[c/x]$	**Ax1**: c free for x in $\sim\psi$
2.	$(\forall x\sim\psi \supset \sim\psi[c/x]) \supset (\psi[c/x] \supset \sim\forall x\sim\psi))$	**Ax0**
3.	$\psi[c/x] \supset \sim\forall x\sim\psi$	MP, 1, 2
4.	$\forall x\psi \supset \psi[c/x]$	**Ax1**: c free for x in ψ
5.	$\forall x\psi$	hyp.
6.	$\psi[c/x]$	MP, 4, 5
7.	$\sim\forall x\sim\psi$	MP, 3, 6

Since tautological instances are all axioms, the following proposition follows immediately by n applications of *modus ponens*. Accordingly, from now on we freely employ purely propositional steps in proofs and justify them by reference to "Proposition T."

60. Proposition T: If $\Gamma \vdash \varphi_1, \ldots, \Gamma \vdash \varphi_n$, and $\varphi_1 \supset (\varphi_2 \supset \cdots \supset (\varphi_n \supset \psi) \cdots)$ is a tautological instance, then $\Gamma \vdash \psi$.

Proof. The formula $\varphi_1 \supset (\varphi_2 \supset \cdots \supset (\varphi_n \supset \psi) \cdots)$ is a tautological instance and hence an axiom; n applications of MP give the desired result., it is required to be shown that □

61. Remark: Since the axioms for predicate logic include the substitution instances of all the propositional axioms, and the only rule (*viz.*, MP) is the same, all the propositional proof-theoretic properties such the Deduction Theorem, Cut, Monotony, etc., carry over to the predicate case, as does Definition 26 of consistency and inconsistency.

62. Theorem: (*Generalization*) If $\Gamma \vdash \varphi$ and x is not free in any formula in Γ, then $\Gamma \vdash \forall x \varphi$.

Proof. By induction on Thm_Γ: if φ is an axiom, so is $\forall x \varphi$. If $\varphi \in \Gamma$ then x is not free in φ, so that $\varphi \supset \forall x \varphi$ is an axiom (**Ax3**), and $\Gamma \vdash \forall x \varphi$ by MP. Now suppose φ follows by MP because $\Gamma \vdash \psi$ and $\Gamma \vdash \psi \supset \varphi$. By the inductive hypothesis, $\Gamma \vdash \forall x \psi$ and $\Gamma \vdash \forall x(\psi \supset \varphi)$. But by **Ax2** also

$$\Gamma \vdash \forall x(\psi \supset \varphi) \supset (\forall x \psi \supset \forall x \varphi),$$

and two applications of MP give $\Gamma \vdash \forall x \varphi$ as desired. □

63. Theorem: (*Weak Generalization on Constants*) Let $\varphi[x/c]$ designate the result of substituting x for all occurrences of c in φ. If $\Gamma \vdash \varphi$ and c is a constant not in occurring in Γ, then there is a variable x not in φ such that $\Gamma \vdash \forall x\, \varphi[x/c]$, and the proof does not involve c.

Proof. Let $\varphi_1, \ldots, \varphi_n$ be a proof of φ from Γ, so that $\varphi = \varphi_n$. Pick a variable x not in $\varphi_1, \ldots, \varphi_n$, and consider the new sequence: $\varphi_1[x/c], \ldots, \varphi_n[x/c]$. Such a sequence is a proof of $\varphi[x/c]$ from Γ. In fact, for each $i = 1, \ldots, n$:

- if φ_i is an axiom, then so is $\varphi_i[x/c]$;
- if $\varphi_i \in \Gamma$, then since c is not in Γ, we have $\varphi_i[x/c] = \varphi_i \in \Gamma$;
- if φ_i is obtained from φ_j and $\varphi_j \supset \varphi_i$ then $\varphi_i[x/c]$ follows by MP from $\varphi_j[x/c]$ and $(\varphi_j \supset \varphi_i)[x/c] = \varphi_j[x/c] \supset \varphi_i[x/c]$.

It is clear that the constant c no longer occurs in the new sequence. Now let Γ' comprise those formulas from Γ that appear in the proof of $\varphi[x/c]$; then x is not free in any formula in Γ' and $\Gamma' \vdash \varphi[x/c]$. By Generalization (Theorem 62) $\Gamma' \vdash \forall x\, \varphi[x/c]$, whence by Monotony $\Gamma \vdash \forall x\, \varphi[x/c]$, as desired. □

Our goal is to replace the requirement in Theorem 63 that x does not occur in φ (at all) by the weaker requirements that x is not free in φ and is free for c in φ. Clearly this can be accomplished by a change of bound variable — so that is what we set out to prove first.

64. Lemma: If x is free for c in φ and y is not free in φ, then x is free for y in $\varphi[y/c]$.

Proof. If x is not free for y in $\varphi[{}^{y}/_{c}]$, then some free occurrence of y in $\varphi[{}^{y}/_{c}]$ falls within the scope of a quantifier $\forall x$. But y is not free in φ, by hypothesis, so all such occurrences come from the substitution $[{}^{y}/_{c}]$. So some occurrence of c falls within the scope of $\forall x$, and x is not free for c in φ. □

65. Lemma: (*Change of Bound Variable*) If x and y are not free in φ and they are both free for c in φ, then $\vdash \forall x \varphi[{}^{x}/_{c}] \equiv \forall y \varphi[{}^{y}/_{c}]$, and the proof does not involve c.

Proof. From the hypotheses, y is free for c in φ and x is not free in φ, so by the previous Lemma 64, y is free for x in $\varphi[{}^{x}/_{c}]$. It follows that $\forall x \varphi[{}^{x}/_{c}] \supset \varphi[{}^{x}/_{c}][{}^{y}/_{x}]$ is an axiom (**Ax1**). Since x is not free in φ, we have $\varphi[{}^{x}/_{c}][{}^{y}/_{x}] = \varphi[{}^{y}/_{c}]$, so that

$$\vdash \forall x \varphi[{}^{x}/_{c}] \supset \varphi[{}^{y}/_{c}],$$

and by the Deduction Theorem $\forall x \varphi[{}^{x}/_{c}] \vdash \varphi[{}^{y}/_{c}]$. Since y is not free in φ, it also not free in $\forall x \varphi[{}^{x}/_{c}]$, so that by Generalization $\forall x \varphi[{}^{x}/_{c}] \vdash \forall y \varphi[{}^{y}/_{c}]$, and the Deduction Theorem again gives $\vdash \forall x \varphi[{}^{x}/_{c}] \supset \forall y \varphi[{}^{y}/_{c}]$. The proof of $\vdash \forall y \varphi[{}^{y}/_{c}] \supset \forall x \varphi[{}^{x}/_{c}]$ is perfectly symmetric, so that the conclusion follows by Proposition T. □

66. Theorem: (*Strong Generalization on Constants*) If $\Gamma \vdash \varphi$, the constant c does not occur in Γ, and x is not free in φ but is free for c in φ, then $\Gamma \vdash \forall x\, \varphi[{}^{x}/_{c}]$.

Proof. Since $\Gamma \vdash \varphi$ and c is not in Γ, then by Weak Generalization on Constants there is a variable y not in φ such that $\Gamma \vdash \forall y\, \varphi[{}^{y}/_{c}]$. Since y is not in φ (at all), it is not free in φ and it is also free for c in φ; if moreover (by hypothesis) x is not free in φ and free for c in φ, then the requirements for a change of bound variable are met, so $\vdash \forall x\, \varphi[{}^{x}/_{c}] \equiv \forall y\, \varphi[{}^{y}/_{c}]$, whence $\Gamma \vdash \forall x\, \varphi[{}^{x}/_{c}]$. □

We conclude this section by collecting facts about identity.

67. Proposition: $\Gamma \vdash t \doteq t$, for any term t and set Γ.

Proof. $\forall x(x \doteq x)$ is an axiom, and any term t is free for x in $x \doteq x$. □

68. Proposition: If $\Gamma \vdash (\varphi[{}^{x}/_{z}] \supset \psi[{}^{y}/_{z}])$ and $\Gamma \vdash t \doteq t'$, and both t and t' are free for z in φ, then $\Gamma \vdash (\varphi[{}^{x}/_{z}] \supset \psi[{}^{y}/_{z}])$.

Proof. Pick variables x and y not occurring φ, t, or t' (at all). Then

$$\forall x \forall y (x \doteq y \supset ((\psi[{}^{x}/_{z}] \supset \varphi[{}^{y}/_{z}]) \supset (\varphi[{}^{x}/_{z}] \supset \psi[{}^{y}/_{z}])))$$

is an axiom (**Ax5**), since x and y are free for z in φ. By **Ax1**, Monotony, and MP (twice), $\Gamma \vdash (t \doteq t' \supset ((\varphi[{}^{x}/_{z}] \supset \psi[{}^{y}/_{z}]) \supset (\varphi[{}^{x}/_{z}] \supset \psi[{}^{y}/_{z}]))$, whence the conclusion follows by two further applications of MP. □

Soundness and completeness of predicate logic

69. Proposition: If φ is an axiom of predicate logic, then $\mathfrak{A} \models \varphi[s]$ for each structure $\mathfrak{A}$ and assignment s.

Proof. We first verify that the schemas **Ax0–Ax5** are valid. For instance, here is the case for **Ax1**: suppose t is free for x in φ, and assume $\mathfrak{A} \models \forall x \varphi[s]$. Then by definition of satisfaction, for each $a \in A$, also $\mathfrak{A} \models \varphi[s({}^{a}/_{x})]$, and in particular this holds when $a = \bar{s}(t)$, i.e., $\mathfrak{A} \models \varphi[s({}^{\bar{s}(t)}/_{x})]$. By the Substitution Theorem, $\mathfrak{A} \models \varphi[{}^{t}/_{x}][s]$. This shows that $\mathfrak{A} \models (\forall x \varphi \supset \varphi[{}^{t}/_{x}])[s]$. After verifying the schemas, we see that their universal closures are also valid, by Corollary 52. □

70. Theorem: (*Soundness*) If $\Gamma \vdash \varphi$ then $\Gamma \models \varphi$.

Proof. By induction on theorems. By the previous proposition, all the axioms are valid, and hence if φ is an axiom then $\Gamma \models \varphi$. Similarly if $\varphi \in \Gamma$. And if $\Gamma \vdash \psi$ and $\Gamma \vdash \psi \supset \varphi$ then $\Gamma \models \varphi$. □

71. Theorem: (*Completeness*) If $\Gamma \models \varphi$ then $\Gamma \vdash \varphi$.

3, 6,

This follows as a corollary, in the usual way, from the theorem that if Γ is consistent then Γ is satisfiable (Theorem 80 below). Some more work is needed before we can give the proof. In particular, in order to prove the theorem, assuming that Γ is satisfiable, we will build a structure $\mathfrak{A}$ and an assignment s satisfying Γ. In so doing we need to extend Γ to a *maximally consistent* set Γ^*, in such a way that if a formula $\exists x \varphi$ is in the set, $\varphi[{}^{t}/_{x}]$ is also in the set for some t. This is arranged by making sure that the maximally consistent set Γ^* extending Γ contains all formulas $\exists x \varphi \supset \varphi[{}^{t}/_{x}]$ for some appropriate *witness* t.

72. Lemma: Let $\mathscr{L}$ and $\mathscr{L}'$ be languages of classical predicate logic. If Γ is consistent in $\mathscr{L}$ and $\mathscr{L}'$ is obtained from $\mathscr{L}$ by adding countably many new constants $c_0, c_1, \ldots$, then Γ is consistent in $\mathscr{L}'$.

Proof. Let $\mathscr{L}'$ be obtained by expanding $\mathscr{L}$ as described. If Γ is consistent in $\mathscr{L}$ but not consistent in $\mathscr{L}'$, then for some formula $\theta(c_1, \ldots, c_n) \in \mathscr{L}'$ both $\Gamma \vdash \theta$ and $\Gamma \vdash {\sim}\theta$. By hypothesis the finitely many new constants $c_1, \ldots, c_n$ occurring in θ do not occur in Γ. By Strong Generalization on Constants there are variables $x_1, \ldots, x_n$ not in θ such that the following are provable in $\mathscr{L}$:

$$\Gamma \vdash \forall x_1 \ldots \forall x_n\, \theta[{}^{x_1}/_{c_1}, \ldots, {}^{x_n}/_{c_n}],$$
$$\Gamma \vdash \forall x_1 \ldots \forall x_n {\sim}\theta[{}^{x_1}/_{c_1}, \ldots, {}^{x_n}/_{c_n}].$$

Since in particular the variables $x_1, \ldots, x_n$ are free for the corresponding constants $c_1, \ldots, c_n$, by **Ax1** also

$$\Gamma \vdash \theta[{}^{x_1}/c_1, \ldots, {}^{x_n}/c_n],$$
$$\Gamma \vdash \sim\theta[{}^{x_1}/c_1, \ldots, {}^{x_n}/c_n],$$

and Γ is inconsistent in $\mathscr{L}'$. □

73. Definition: A set Δ of formulas of a language $\mathscr{L}$ is *saturated* if and only if for each formula $\varphi \in \mathscr{L}$ and variable x there is a constant c such that $\sim\forall x\varphi \supset \sim\varphi[{}^{c}/x]$ is in Δ.

74. Definition: Fix an enumeration $\langle\varphi_0, x_0\rangle, \langle\varphi_1, x_1\rangle, \ldots$ of all formula-variable pairs of $\mathscr{L}'$, and define the formula θ_n by recursion on n. For θ_0 let c_0 be the first new constant that does not occur in φ_0 and let θ_0 be the formula $\sim\forall x_0\varphi_0 \supset \sim\varphi_0[{}^{c_0}/x_0]$. Assuming $\theta_0, \ldots, \theta_n$ have been defined, denote by c_{n+1} the first new constant not occurring in $\theta_0, \ldots \theta_n$ or φ_{n+1}, and let θ_{n+1} be the formula: $\sim\forall x_{n+1}\varphi_{n+1} \supset \sim\varphi_{n+1}[{}^{c_{n+1}}/x_{n+1}]$. Finally, put $\Theta = \{\theta_n : n \geq 0\}$.

75. Proposition: (*Saturation*) Every consistent set Γ can be extended to a saturated consistent set.

Proof. Given a consistent Γ, expand the language by adding countably many new constants. By Lemma 72, Γ is still consistent in the richer language. Further, let Θ be as in Definition 74; then $\Gamma \cup \Theta$ is saturated by construction. To show that it is also consistent it suffices to show, by induction on n, that each set of the form $\Gamma \cup \{\theta_0, \ldots \theta_n\}$ is consistent.

For the basis of the induction, suppose that $\Gamma \cup \{\theta_0\}$ is inconsistent. It follows by Proposition 28 that $\Gamma \vdash \sim\theta_0$, whence both the following hold by Proposition T:

$$\Gamma \vdash \sim\forall x_0\varphi_0, \qquad\qquad \Gamma \vdash \varphi_0[{}^{c_0}/x_0].$$

But c_0 does not occur in Γ and the variable x_0 is not free in $\varphi_0[{}^{c_0}/x_0]$. Moreover, x_0 is free for c_0 in $\varphi_0[{}^{c_0}/x_0]$ (since by choice of c_0 the only occurrences of c_0 in $\varphi_0[{}^{c_0}/x_0]$ derive from the substitution $[{}^{c_0}/x_0]$ and therefore fall outside the scope of any quantifier binding x_0). By Strong Generalization on Constants, from the latter of these we obtain $\Gamma \vdash \forall x_0\varphi_0[{}^{c_0}/x_0][{}^{x_0}/c_0]$, i.e., $\Gamma \vdash \forall x_0\varphi_0$, and Γ itself is inconsistent. The inductive step is perfectly analogous. □

76. Proposition: Every consistent set Γ can be extended to a maximally consistent saturated set Γ^*.

Proof. Let Γ be consistent, and Θ as in Definition 74. By proposition 75, $\Gamma \cup \Theta$ is a consistent saturated set in the richer language with the countably many new constants. We define Γ^* exactly as in propositional logic: let $\varphi_0, \varphi_1, \ldots$ be an enumeration of all the formulas of $\mathscr{L}'$. Define $\Gamma_0 = \Gamma \cup \Theta$ and

$$\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{\varphi_n\} & \text{if } \Gamma_n \cup \{\varphi_n\} \text{ is consistent;} \\ \Gamma_n \cup \{\sim\varphi_n\} & \text{otherwise.} \end{cases}$$

Putting $\Gamma^* = \bigcup_{n\geq 0} \Gamma_n$, we obtain a maximally consistent set in $\mathscr{L}'$; since $\Gamma \cup \Theta$ is saturated, so is Γ^*. $\square$

Our plan is to extract from Γ^* a structure $\mathfrak{A}'$ satisfying Γ^* in $\mathscr{L}'$, which can then be turned into a structure $\mathfrak{A}$ satisfying Γ in $\mathscr{L}$ simply by "forgetting" the interpretation of the new constants c_n. However, if the universe A' of $\mathfrak{A}'$ is simply taken to be the set of all closed terms, this leads to a problem, in that we might have $\Gamma^* \vdash t_1 \doteq t_2$ for *distinct* terms t_1 and t_2. This motivates the following definition of an equivalence relation and the corresponding detour through the resulting quotient.

77. Definition: Define a relation $\approx$ over the set of terms of $\mathscr{L}'$ by setting $t \approx t'$ if and only $\Gamma^* \vdash t \doteq t'$.

78. Lemma: The relation $\approx$ is an equivalence relation over the terms of $\mathscr{L}'$.

Proof. Reflexivity is given by Proposition 67; symmetry and transitivity follow, in a similar way to Proposition 68. For instance, symmetry: by **Ax5** and **Ax1**, $\Gamma^* \vdash t \doteq t' \supset (t \doteq t \supset t' \doteq t)$, so that if $t \approx t'$ then $\Gamma^* \vdash t \doteq t'$; but also $\Gamma^* \vdash t \doteq t$, as we just saw, so $\Gamma^* \vdash t' \doteq t$, i.e., $t' \approx t$. $\square$

79. Definition: For each term t of $\mathscr{L}'$, the equivalence class $\{t' : t \approx t'\}$ of t relative to $\approx$ is denoted by $[t]$.

We are now ready to complete the proof of the main theorem.

80. Theorem: If Γ is consistent then Γ is satisfiable.

Proof. Given a consistent Γ, we extend it to a maximally consistent saturated set Γ^* of formulas of $\mathscr{L}'$, using Proposition 76. We define a structure $\mathfrak{A}'$ for $\mathscr{L}'$ and an assignment s as follows:

- $|\mathfrak{A}'| = A' = \{[t] : t \in \mathscr{L}'\}$;
- $c^{\mathfrak{A}'} = [c]$ for each constant c of $\mathscr{L}'$.
- $\langle [t_1], \ldots, [t_n] \rangle \in P^{\mathfrak{A}'}$ if and only if $\Gamma^* \vdash P t_1 \ldots t_n$, for each n-place predicate symbol P;
- $f^{\mathfrak{A}'}([t_1], \ldots, [t_n]) = [f t_1 \ldots t_n]$, for each n-place function symbol f;
- $s(x) = [x]$.

It is important to notice that the definitions3, 6, of $P^{\mathfrak{A}'}$ and $f^{\mathfrak{A}'}$ are "independent of the representatives," since the relation $\approx$ is a congruence with respect to $P^{\mathfrak{A}'}$ and $f^{\mathfrak{A}'}$. For instance, if $t \approx t'$ then $\Gamma^* \vdash t \doteq t'$ whence also $\Gamma^* \vdash f t_1 \ldots t \ldots t_n \doteq f t_1 \ldots t' \ldots t_n$ so that $f t_1 \ldots t \ldots t_n \approx f t_1 \ldots t' \ldots t_n$. It follows that $f^{\mathfrak{A}'}([t_1], \ldots, [t], \ldots, [t_n]) = f^{\mathfrak{A}'}([t_1], \ldots, [t'], \ldots, [t_n])$ and the definition is independent of the representatives.

Next, one easily shows that for each term t, $\bar{s}(t) = [t]$ (by induction on t). Similarly, we prove that $\mathfrak{A}' \models \varphi[s]$ if and only if $\varphi \in \Gamma^*$, whence in particular s satisfies Γ^*. The proof that $\mathfrak{A}' \models \varphi[s]$ if and only if $\varphi \in \Gamma^*$ is by induction on φ, the crucial case being the one for the universal quantifier:

- If $\mathfrak{A}' \not\models \forall x \psi[s]$, then for some term t, $\mathfrak{A}' \not\models \psi[s([t]/x)]$, i.e., $\mathfrak{A}' \not\models \psi[s(\bar{s}(t)/x)]$. Using Lemma 65 (Change of Bound Variable), successively rename each quantifier $\forall y$ in ψ as needed to obtain a formula ψ' such that t is free for x in ψ' and $\vdash \psi \equiv \psi'$. It follows that $\mathfrak{A}' \not\models \psi'[s(\bar{s}(t)/x)]$, and by the Substitution Theorem (Theorem 54), also $\mathfrak{A}' \not\models \psi'[t/x][s]$. By the inductive hypothesis, $\psi'[t/x] \notin \Gamma^*$, whence $\forall x \psi' \notin \Gamma^*$. But by deductive closure and $\vdash \psi \equiv \psi'$, we have $\forall x \psi \notin \Gamma^*$.

- If $\forall x \psi \notin \Gamma^*$, then by maximality $\sim\forall x \psi \in \Gamma^*$ (i.e., $\exists x \sim \psi \in \Gamma$), and by saturation for some c, also $\sim\psi[c/x] \in \Gamma^*$, so that by consistency $\psi[c/x] \notin \Gamma^*$. (The constant c is the "existential witness" referred to in the Introduction.) By the inductive hypothesis, $\mathfrak{A}' \not\models \psi[c/x][s]$, so that $\mathfrak{A}' \not\models \psi[s(\bar{s}(c)/x)]$ by the Substitution Theorem. The conclusion $\mathfrak{A}' \not\models \forall x \psi[s]$ follows.

3, 6,

Finally, let $\mathfrak{A}$ be the structure for $\mathscr{L}$ obtained from $\mathfrak{A}'$ by dropping the interpretations of the new constants c_n. Using induction on formulas φ of $\mathscr{L}$ one can show that $\mathfrak{A}' \models \varphi[s]$ if and only if $\mathfrak{A} \models \varphi[s]$, so that s satisfies Γ in $\mathfrak{A}$. □

81. Definition: A set Γ of formulas is *finitely satisfiable* if and only if every finite $\Gamma_0 \subseteq \Gamma$ is satisfiable.

82. Corollary: (*Compactness Theorem*)

(*i*) if $\Gamma \models \varphi$ then there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \models \varphi$;

(*ii*) Γ is satisfiable if and only if it is finitely satisfiable.

Part IV

Extensions and applications

Rudiments of model theory

83. Definition: A *signature* $\mathfrak{s}$ is a set of predicate symbols, function symbols and individual constants. A structure $\mathfrak{A}$ for $\mathfrak{s}$ assigns subsets of A^n, functions from A^n to A, and members of A to the predicate symbols, function symbols and individual constants, respectively, in the signature.

84. Definition: Given signatures $\mathfrak{s} \subseteq \mathfrak{t}$ and a structure $\mathfrak{A}$ for $\mathfrak{t}$, the *reduct* of $\mathfrak{A}$ to $\mathfrak{s}$ is the structure $\mathfrak{B}$ obtained from $\mathfrak{A}$ by dropping interpretations for the symbols in $\mathfrak{t}$ but not in $\mathfrak{s}$. Similarly, we say that $\mathfrak{A}$ is an *expansion* of $\mathfrak{B}$ to $\mathfrak{t}$.

85. Definition: Given structures $\mathfrak{A}$ and $\mathfrak{B}$ for the same signature, we say that $\mathfrak{A}$ is a *substructure* of $\mathfrak{B}$, and $\mathfrak{B}$ an *extension* of $\mathfrak{A}$, written $\mathfrak{A} \subseteq \mathfrak{B}$, if $A \subseteq B$ and moreover:

- For each constant c in the signature, $c^{\mathfrak{A}} = c^{\mathfrak{B}}$;
- For each predicate symbol P and $a_1, \ldots, a_n$ in A, $P^{\mathfrak{A}}(a_1, \ldots, a_n)$ holds if and only if $P^{\mathfrak{B}}(a_1, \ldots, a_n)$ holds.
- For each function symbol f in the signature and $a_1, \ldots a_n$ in A, $f^{\mathfrak{A}}(a_1, \ldots a_n) = f^{\mathfrak{B}}(a_1, \ldots, a_n)$.

86. Remark: If the signature contains no constant or function symbols, then each $A \subseteq B$ determines a substructure $\mathfrak{A}$ of $\mathfrak{B}$ with universe A by putting $P^{\mathfrak{A}} = P^{\mathfrak{B}} \cap A^n$.

87. Definition: If a structure $\mathfrak{A}$ satisfies a set Γ of sentences, then we say that $\mathfrak{A}$ is a *model* of Γ.

88. Proposition: (*Downward Löwenheim-Skolem Theorem*) If Γ is consistent then it has a countable model, i.e., it is satisfiable in a structure whose domain is either finite or denumerably infinite.

Proof. If Γ is consistent, the structure $\mathfrak{A}$ delivered by the proof of Theorem 80 has a universe A whose cardinality is bounded by that of the set of the terms of the language $\mathscr{L}'$. So A is at most denumerably infinite. □

89. Remark: (*Skolem's paradox*) By the Löwenheim-Skolem theorem, Zermelo-Fränkel set theory, ZF, is satisfiable in a countable structure, and yet $ZF \vdash \exists x(x$ is uncountable).

90. Theorem: If a set Γ of sentences has arbitrarily large finite models, then it has an infinite model.

Proof. Expand the language of Γ by adjoining countably many new constants $c_0, c_1, \ldots$ and consider the set $\Gamma \cup \{\sim c_i \doteq c_j : i \neq j\}$. To say that Γ has arbitrarily large finite models means that for every $m > 0$ there is an $n \geq m$ such that Γ has a model of cardinality n. This implies that $\Gamma \cup \{\sim c_i \doteq c_j : i \neq j\}$ is finitely satisfiable. By compactness, $\Gamma \cup \{\sim c_i \doteq c_j : i \neq j\}$ has a model $\mathfrak{A}$ whose domain must be infinite, since it satisfies all inequalities $\sim c_i \doteq c_j$. □

91. Proposition: There is no sentence φ of any first-order language $\mathscr{L}$ that is true in a structure $\mathfrak{A}$ if and only if the domain A of the structure is infinite.

Proof. If there were such a φ, its negation $\sim\varphi$ would be true in all and only the finite structures, and it would therefore have arbitrarily large finite models but it would lack an infinite model, against Theorem 90. □

92. Definition: Given two structures $\mathfrak{A}$ and $\mathfrak{B}$ for the same language $\mathscr{L}(\mathfrak{s})$, we say that $\mathfrak{A}$ is *elementarily equivalent to* $\mathfrak{B}$, written $\mathfrak{A} \equiv \mathfrak{B}$, if and only if for every *sentence* φ of $\mathscr{L}(\mathfrak{s})$, $\mathfrak{A} \models \varphi$ if and only if $\mathfrak{B} \models \varphi$.

93. Definition: Given two structures $\mathfrak{A}$ and $\mathfrak{B}$ for the same language $\mathscr{L}$, we say that $\mathfrak{A}$ is *isomorphic to* $\mathfrak{B}$, written $\mathfrak{A} \simeq \mathfrak{B}$, if and only if there is a function $\mathsf{h} : A \to B$ such that:

1. h is one-one: if $\mathsf{h}(a_1) = \mathsf{h}(a_2)$ then $a_1 = a_2$;
2. h is onto B: for every $b \in B$ there is $a \in A$ such that $\mathsf{h}(a) = b$;
3. for every constant c: $\mathsf{h}(c^{\mathfrak{A}}) = c^{\mathfrak{B}}$;
4. for every n-place predicate symbol P: $\langle a_1, \ldots, a_n \rangle \in P^{\mathfrak{A}}$ if and only if $\langle \mathsf{h}(a_1), \ldots, \mathsf{h}(a_n) \rangle \in P^{\mathfrak{B}}$;
5. for every n-place function symbol f: $\mathsf{h}(f^{\mathfrak{A}}(a_1, \ldots, a_n)) = f^{\mathfrak{B}}(\mathsf{h}(a_1), \ldots, \mathsf{h}(a_n))$.

94. Theorem: If $\mathfrak{A} \simeq \mathfrak{B}$ then $\mathfrak{A} \equiv \mathfrak{B}$.

Proof. Let h be an isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$; for any assignment s in $\mathfrak{A}$, let $\mathsf{h} \circ s$ be the composition of h and s, i.e., the assignment in $\mathfrak{B}$ such that $(\mathsf{h} \circ s)(x) = \mathsf{h}(s(x))$. We proceed by induction on t and φ and prove the stronger claims:

$$\mathsf{h}(\bar{s}(t)) = \overline{\mathsf{h} \circ s}(t);$$

$$\mathfrak{A} \models \varphi[s] \text{ if and only if } \mathfrak{B} \models \varphi[\mathsf{h} \circ s].$$

Make sure to take note at each step of how each of the five properties characterizing isomorphisms is used. □

95. Definition: Given a structure $\mathfrak{A}$, the *theory* of $\mathfrak{A}$ is the set $\mathsf{Th}(\mathfrak{A})$ of sentences that are true in $\mathfrak{A}$, i.e.: $\mathsf{Th}(\mathfrak{A}) = \{\varphi : \mathfrak{A} \models \varphi\}$.

We also use the term "theory" informally to refer to sets of sentences having an intended interpretation, whether deductively closed or not.

96. Proposition: For any $\mathfrak{A}$, $\mathsf{Th}(\mathfrak{A})$ is maximally consistent. Hence, if $\mathfrak{B} \models \psi$ for every $\psi \in \mathsf{Th}(\mathfrak{A})$, then $\mathfrak{A} \equiv \mathfrak{B}$.

Proof. $\mathsf{Th}(\mathfrak{A})$ is consistent because satisfiable (by definition). It is maximal since for any *sentence* φ either φ is true in $\mathfrak{A}$ or its negation is. It immediately follows that $\mathsf{Th}(\mathfrak{A}) \subseteq \mathsf{Th}(\mathfrak{B})$ and $\mathsf{Th}(\mathfrak{B}) \subseteq \mathsf{Th}(\mathfrak{A})$, whence $\mathfrak{A} \equiv \mathfrak{B}$. □

97. Remark: Consider $\mathfrak{R} = (\mathbb{R}, <)$, the structure whose domain is the set $\mathbb{R}$ of the real numbers, in the language comprising only a 2-place predicate interpreted as the $<$ relation over the reals. Clearly $\mathfrak{R}$ is uncountable; however, since $\mathsf{Th}(\mathfrak{R})$ is obviously consistent, by the Löwenheim-Skolem theorem it has a countable model, say $\mathfrak{S}$, and by Proposition 96, $\mathfrak{R} \equiv \mathfrak{S}$. Moreover, since $\mathfrak{R}$ and $\mathfrak{S}$ are not isomorphic, this shows that the converse of Theorem 94 fails in general.

98. Definition: Given two structures $\mathfrak{A}$ and $\mathfrak{B}$, a *partial isomorphism* from $\mathfrak{A}$ to $\mathfrak{B}$ is a finite function p taking arguments in A and returning values in B, satisfying the isomorphism conditions from Definition 93 on its domain:

1. p is one-one;
2. for every constant c: if $c^{\mathfrak{A}}$ is in the domain of p, then $p(c^{\mathfrak{A}}) = c^{\mathfrak{B}}$;
3. for every n-place predicate symbol P: if $a_1, \dots, a_n$ are in the domain of p, then $\langle a_1, \dots, a_n \rangle \in P^{\mathfrak{A}}$ if and only if $\langle p(a_1), \dots, p(a_n) \rangle \in P^{\mathfrak{B}}$;
4. for every n-place function symbol f: if $a_1, \dots, a_n$ are in the domain of p, then $p(f^{\mathfrak{A}}(a_1, \dots, a_n))$ is also in the domain of p, and $p(f^{\mathfrak{A}}(a_1, \dots, a_n)) = f^{\mathfrak{B}}(p(a_1), \dots, p(a_n))$.

Notice that the empty map $\varnothing$ is a partial isomorphism between any two structures.

99. Definition: Two structures $\mathfrak{A}$ and $\mathfrak{B}$, are *partially isomorphic*, written $\mathfrak{A} \simeq_p \mathfrak{B}$, if and only if there is a non-empty set $\mathscr{I}$ of partial isomorphisms between $\mathfrak{A}$ and $\mathfrak{B}$ satisfying the *back-and-forth* property:

1. (*Forth*) For every $p \in \mathscr{I}$ and $a \in A$ there is $q \in \mathscr{I}$ such that $p \subseteq q$ and a is in the domain of q;
2. (*Back*) For every $p \in \mathscr{I}$ and $b \in B$ there is $q \in \mathscr{I}$ such that $p \subseteq q$ and b is in the range of q.

100. Theorem: If $\mathfrak{A} \simeq_p \mathfrak{B}$ and $\mathfrak{A}$ and $\mathfrak{B}$ are countable, then $\mathfrak{A} \simeq \mathfrak{B}$.

Proof. Since $\mathfrak{A}$ and $\mathfrak{B}$ are countable, let $A = \{a_0, a_1, \dots\}$ and $B = \{b_0, b_1, \dots\}$. Starting with an arbitrary $p_0 \in \mathscr{I}$, we define an increasing sequence of partial isomorphisms $p_0 \subseteq p_1 \subseteq p_2 \subseteq \cdots$ as follows:

- ► if $n+1$ is odd, say $n = 2r$, then using the Forth property find a $p_{n+1} \in \mathscr{I}$ such that $p_n \subseteq p_{n+1}$ and a_r is in the domain of p_{n+1};
- ► if $n+1$ is even, say $n+1 = 2r$, then using the Back property find a $p_{n+1} \in \mathscr{I}$ such that $p_n \subseteq p_{n+1}$ and b_r is in the range of p_{n+1}.

If we now put:

$$\mathsf{p} = \bigcup_{n \geq 0} p_n,$$

We have that p is a an isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$. □

101. Theorem: If $\mathfrak{A}$ and $\mathfrak{B}$ are structures in a purely relational signature (a signature containing only predicate symbols, and no function symbols or constants), then $\mathfrak{A} \simeq_p \mathfrak{B}$, implies $\mathfrak{A} \equiv \mathfrak{B}$.

Proof. By induction on formulas, one shows that if $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are such that there is a partial isomorphism p mapping each a_i to b_i and $s_1(x_i) = a_i$ and $s_2(x_i) = b_i$ (for $i = 1, \ldots, n$), then $\mathfrak{A} \models \varphi[s_1]$ if and only if $\mathfrak{B} \models \varphi[s_2]$. The case for $n = 0$ gives $\mathfrak{A} \equiv \mathfrak{B}$. □

If function symbols are present, the previous result is still true, but one needs to consider the isomorphism induced by p between the substructure of $\mathfrak{A}$ generated by $a_1, \ldots, a_n$ and the substructure of $\mathfrak{B}$ generated by $b_1, \ldots, b_n$. But we will not need this more general case in what follows.

The previous result can be "broken down" in stages by establishing a connection between the number of nested quantifiers in a formula and how many times the relevant partial isomorphisms can be extended.

102. Definition: For any formula φ, the quantifier rank of φ, denoted by $\mathsf{qr}(\varphi) \in \mathbb{N}$, is recursively defined as the highest number of nested quantifiers in φ. Two structures $\mathfrak{A}$ and $\mathfrak{B}$ are n-equivalent, written $\mathfrak{A} \equiv_n \mathfrak{B}$, if they agree on all sentences of quantifier rank less than or equal to n.

103. Proposition: Let $\mathfrak{s}$ be a finite purely relational signature, i.e., a signature containing finitely many predicate and constant symbols, and no function symbols. Then for each $n \in \mathbb{N}$ there are only finitely many first-order sentences in the signature $\mathfrak{s}$ that have quantifier rank no greater than n, up to logical equivalence.

Proof. By induction on n. □

104. Definition: Given a structure $\mathfrak{A}$, let $A^{<\omega}$ be the set of all finite sequences over A. We use variables $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$ to range over finite sequences of elements. If $\mathbf{a} \in A^{<\omega}$ and $a \in A$, then $\mathbf{a}a$ represents the concatenation of $\mathbf{a}$ with a.

105. Definition: Given structures $\mathfrak{A}$ and $\mathfrak{B}$, we define relations $I_n \subseteq A^{<\omega} \times B^{<\omega}$ between sequences of equal length, by recursion on n as follows:

- ► $I_0(\mathbf{a}, \mathbf{b})$ if and only if $\mathbf{a}$ and $\mathbf{b}$ satisfy the same atomic formulas in $\mathfrak{A}$ and $\mathfrak{B}$; i.e., if $s_1(x_i) = a_i$ and $s_2(x_i) = b_i$ and φ is atomic with all variables among $x_1, \ldots, x_n$, then $\mathfrak{A} \models \varphi[s_1]$ if and only if $\mathfrak{B} \models \varphi[s_2]$.

► $I_{n+1}(\mathbf{a},\mathbf{b})$ if and only if for every $a \in A$ there is a $b \in B$ such that $I_n(\mathbf{a}a,\mathbf{b}b)$, and vice-versa.

106. Definition: Write $\mathfrak{A} \approx_n \mathfrak{B}$ if $I_n(\emptyset,\emptyset)$ holds of $\mathfrak{A}$ and $\mathfrak{B}$ (where $\emptyset$ is the empty sequence).

107. Theorem: Let $\mathfrak{s}$ be a purely relational signature. Then $I_n(\mathbf{a},\mathbf{b})$ implies that for every φ such that $\mathsf{qr}(\varphi) \leq n$, we have $\mathfrak{A} \models \varphi[\mathbf{a}]$ if and only if $\mathfrak{B} \models \varphi[\mathbf{b}]$ (where again $\mathbf{a}$ satisfies φ if any s such that $s(x_i) = a_i$ satisfies φ). Moreover, if $\mathfrak{s}$ is finite, the converse also holds.

Proof. The proof that $I_n(\mathbf{a},\mathbf{b})$ implies that $\mathbf{a}$ and $\mathbf{b}$ satisfy the same formulas of quantifier rank no greater than n is by an easy induction on φ. For the converse we proceed by induction on n, using Proposition 103, which ensures that for each n there are at most finitely many non-equivalent formulas of that quantifier rank.

For $n = 0$ the hypothesis that $\mathbf{a}$ and $\mathbf{b}$ satisfy the same quantifier-free formulas gives that they satisfy the same atomics, so that $I_0(\mathbf{a},\mathbf{b})$.

For the $n+1$ case, suppose that $\mathbf{a}$ and $\mathbf{b}$ satisfy the same formulas of quantifier rank no greater than $n+1$; in order to show that $I_{n+1}(\mathbf{a},\mathbf{b})$, it suffices to show that for each $a \in A$ there is a $b \in B$ such that $I_n(\mathbf{a}a,\mathbf{b}b)$, and by the inductive hypothesis again it suffices to show that for each $a \in A$ there is a $b \in B$ such that $\mathbf{a}a$ and $\mathbf{b}b$ satisfy the same formulas of quantifier rank no greater than n.

Given $a \in A$, let τ_n^a be set of formulas $\psi(x,\mathbf{y})$ of rank no greater than n satisfied by $\mathbf{a}a$ in $\mathfrak{A}$; τ_n^a is finite, so we can assume it is a single first-order formula. It follows that $\mathbf{a}$ satisfies $\exists x \tau_n^a(x,\mathbf{y})$, which has quantifier rank no greater than $n+1$. By hypothesis $\mathbf{b}$ satisfies the same formula in $\mathfrak{B}$, so that there is a $b \in B$ such that $\mathbf{b}b$ satisfies τ_n^a; in particular, $\mathbf{b}b$ satisfies the same formulas of quantifier rank no greater than n as $\mathbf{a}a$. Similarly one shows that for every $b \in B$ there is $a \in A$ such that $\mathbf{a}a$ and $\mathbf{b}b$ satisfy the same formulas of quantifier rank no greater than n, which completes the proof. □

108. Corollary: If $\mathfrak{A}$ and $\mathfrak{B}$ are purely relational structures in a finite signature then $\mathfrak{A} \approx_n \mathfrak{B}$ if and only if $\mathfrak{A} \equiv_n \mathfrak{B}$. In particular $\mathfrak{A} \equiv \mathfrak{B}$ if and only if for each n, $\mathfrak{A} \approx_n \mathfrak{B}$.

109. Definition: A *dense linear ordering without endpoints* is a structure $\mathfrak{A} = (A, <^{\mathfrak{A}})$ satisfying the following sentences:

1. $\forall x \sim x < x$;
2. $\forall x \forall y \forall z (x < y \supset (y < z \supset x < z))$;
3. $\forall x \forall y (x < y \lor x \doteq y \lor y < x)$;
4. $\forall x \exists y (x < y)$;
5. $\forall x \exists y (y < x)$;
6. $\forall x \forall y (x < y \supset \exists z (x < z \land z < y))$.

110. Theorem: (*Cantor*) Any two countable dense linear orderings without endpoints are isomorphic.

Proof. Let $\mathfrak{A} = (A, <^{\mathfrak{A}})$ and $\mathfrak{B} = (B, <^{\mathfrak{B}})$ be countable dense linear orderings without endpoints, and let $\mathscr{I}$ be the set of all partial isomorphisms between them. $\mathscr{I}$ is not empty since at least $\emptyset \in \mathscr{I}$. It suffices to show that $\mathscr{I}$ satisfies the Back-and-Forth property.

Let $p \in \mathscr{I}$ and let $p(a_i) = b_i$ for $i = 1, \ldots, n$, and without loss of generality suppose $a_1 <^{\mathfrak{A}} a_2 <^{\mathfrak{A}} \cdots <^{\mathfrak{A}} a_n$. Given $a \in A$, find $b \in B$ as follows:

- if $a <^{\mathfrak{A}} a_1$ let $b \in B$ be such that $b <^{\mathfrak{B}} b_1$;
- if $a_n <^{\mathfrak{A}} a$ let $b \in B$ be such that $b_n <^{\mathfrak{B}} b$;
- if $a_i <^{\mathfrak{A}} a <^{\mathfrak{A}} a_{i+1}$ for some i, then let $b \in B$ be such that $b_i <^{\mathfrak{B}} b <^{\mathfrak{B}} b_{i+1}$.

It is always possible to find a b with the desired property since $\mathfrak{B}$ is a dense linear ordering without endpoints. Define $q = p \cup \{\langle a, b \rangle\}$, so that $q \in \mathscr{I}$ is the desired extension of p. This establishes the Forth property. The Back property is similar. So $\mathfrak{A} \simeq_p \mathfrak{B}$ and by Theorem 100 also $\mathfrak{A} \simeq \mathfrak{B}$. □

111. Remark: Let $\mathfrak{S} = (S, <^{\mathfrak{S}})$ be any countable dense linear ordering without endpoints. Then (by Theorem 110) $\mathfrak{S} \simeq \mathfrak{Q}$, where $\mathfrak{Q} = (\mathbb{Q}, <)$ is the countable dense linear ordering having the set $\mathbb{Q}$ of the rational numbers as its domain. Now consider again the structure $\mathfrak{R} = (\mathbb{R}, <)$ from Remark 97. We saw that there is a countable structure $\mathfrak{S}$ such that $\mathfrak{R} \equiv \mathfrak{S}$. But $\mathfrak{S}$ is a countable dense linear ordering without endpoints, and so it is isomorphic (and hence equivalent) to the structure $\mathfrak{Q}$. By transitivity of elementary equivalence, $\mathfrak{R} \equiv \mathfrak{Q}$. (We could have shown this directly by establishing $\mathfrak{R} \simeq_p \mathfrak{Q}$ by the same back-and-forth argument.)

Non-standard models of arithmetic

112. Definition: Let $\mathscr{L}_N$ be the language of arithmetic, comprising a constant $\mathbf{0}$, a 2-place predicate symbol $<$, a 1-place function symbol s, and 2-place function symbols $+$ and $\times$.

- The *standard model* of arithmetic is the structure $\mathfrak{N}$ for $\mathscr{L}_N$ having $\mathbb{N} = \{0, 1, 2, \ldots\}$ and interpreting $\mathbf{0}$ as 0, $<$ as the less-than relation over $\mathbb{N}$, and s, $+$ and $\times$ as successor, addition, and multiplication over $\mathbb{N}$, respectively.
- *True arithmetic* is the theory $\mathsf{Th}(\mathfrak{N})$.

When working in $\mathscr{L}_n$ we abbreviate each term of the form $s \cdots s\mathbf{0}$, with n applications of the successor function to $\mathbf{0}$, as $\mathbf{n}$.

113. Definition: A structure $\mathfrak{M}$ for $\mathscr{L}_N$ is *standard* if and only $\mathfrak{N} \simeq \mathfrak{M}$.

114. Theorem: There are non-standard countable models of true arithmetic.

Proof. Expand $\mathscr{L}_N$ by introducing a new constant, c, and consider the theory

$$\mathsf{Th}(\mathfrak{N}) \cup \{\mathbf{n} < c : n \in \mathbb{N}\}.$$

The theory is finitely satisfiable, so by compactness it has a model $\mathfrak{M}$, which can be taken to be countable by the Downward Löwenheim-Skolem theorem. Where M is the domain of $\mathfrak{M}$, let $\mathfrak{M}$ interpret the non-logical constants of $\mathscr{L}$ as $\mathbf{z} \in M$, $\prec \subseteq M^2$, $* : M \to M$, and $\oplus, \otimes : M^2 \to M$. For each $x \in M$, we write x^* for the element of M obtained from x by application of $*$.

Now, if h were an isomorphism of $\mathfrak{N}$ and $\mathfrak{M}$, there would be $n \in \mathbb{N}$ such that $\mathsf{h}(n) = c^{\mathfrak{M}}$. So let s be any assignment in $\mathfrak{N}$ such that $s(x) = n$ (we use s both for the successor symbol in $\mathcal{L}_N$ and the assignment: no confusion should arise). Then $\mathfrak{N} \models \mathbf{n} \doteq x[s]$; by the proof of Theorem 94, also $\mathfrak{M} \models \mathbf{n} \doteq x[\mathsf{h} \circ s]$, so that $c^{\mathfrak{M}} = \mathbf{z}^{*\cdots *}$ (with $*$ iterated n times). But this is impossible since by assumption $\mathfrak{M} \models \mathbf{n} < c$ and $\prec$ is irreflexive. So $\mathfrak{M}$ is non-standard. □

Since the non-standard model $\mathfrak{M}$ from Theorem 114 is elementarily equivalent to the standard one, a number of properties of $\mathfrak{M}$ can be derived. The rest of this section is devoted to such a task, which will allow us to obtain a precise characterization of countable non-standard models of $\mathsf{Th}(\mathfrak{N})$.

1. No member of M is $\prec$-less than itself: the sentence $\forall x \sim x < x$ is true in $\mathfrak{N}$ and therefore in $\mathfrak{M}$.
2. By a similar reasoning we obtain that $\prec$ is a *linear ordering* of M, i.e., a total, irreflexive, transitive relation on M.
3. The element $\mathbf{z}$ is the $\prec$-least element of M.
4. Any member of M is $\prec$-less than its $*$-successor and x^* is the $\prec$-least member of M greater than x.
5. $\mathfrak{M}$ contains an initial segment (of $\prec$) isomorphic to $\mathbb{N}$: $\mathbf{z}, \mathbf{z}^*, \mathbf{z}^{**}, \ldots$, which we call the *standard part* of M. Any other member of M is *non-standard*. There must be non-standard members of M, or else the function h from the proof of Theorem 114 is an isomorphism. We use $n, m, \ldots$ as variables ranging on this standard part of $\mathfrak{M}$.
6. Every non-standard element is greater than any standard one; this is because for every $n \in \mathbb{N}$,

   $$\mathfrak{N} \models \forall z(\sim(z \doteq \mathbf{0} \lor \ldots \lor z \doteq \mathbf{n}) \supset \mathbf{n} < z),$$

   so if $z \in M$ is different from al the standard elements, it must be *greater* than all of them.
7. Any member of M other than $\mathbf{z}$ is the $*$-successor of some unique element of M, denoted by ${}^*x$. If $x = y^*$ then both x and y are standard if one of them is (and both non-standard if one of them is).
8. Define an equivalence relation $\approx$ over M by saying that $x \approx y$ if and only if for some *standard* n, either $x \oplus n = y$ or $y \oplus n = x$. In other words, $x \approx y$ if and only if x and y are a finite distance apart. If n and m are standard then $n \approx m$. Define the *block* of x to be the equivalence class $[x] = \{y : x \approx y\}$.
9. Suppose that $x \prec y$ where $x \not\approx y$; then either $x^* \prec y$ or $x^* = y$; the latter is impossible because it implies $x \approx y$, so $x \prec y$. Similarly, if $x \prec y$ and $x \not\approx y$, then $x \prec {}^*y$. Therefore if $x \prec y$ and $x \not\approx y$, then every $w \approx x$ is $\prec$-less than every $v \approx y$. Accordingly, each block $[x]$ forms a doubly infinite chain

   $$\cdots \prec {}^{**}x \prec {}^*x \prec x \prec x^* \prec x^{**} \prec \cdots$$

   which is referred to as a Z-chain because it has the order type of the integers.
10. The $\prec$ ordering can be lifted up the blocks: if $x \prec y$ then the block of x is less than the block of

y. A block is *non-standard* if it contains a non-standard element. The standard block is the least block.

11. There is no least non-standard block: if y is non-standard then there is a $x \prec y$ where x is also non-standard and $x \not\approx y$. Proof: in the standard model $\mathfrak{N}$, every number is divisible by two, possibly with remainder one. By elementary equivalence, for every $y \in M$ there is $x \in M$ such that either $x \oplus x = y$ or $x \oplus x \oplus \mathbf{z}^* = y$. If x were standard, then so would be y; so x is non-standard. And $x \not\approx y$ for if $x \oplus n = y$ for some standard n then (say) $x \oplus n = x \oplus x$, whence $x = n$ by the cancellation law for addition (which holds in $\mathfrak{N}$ and therefore in $\mathfrak{M}$ as well), and x would be standard after all. (Similarly if $x \oplus x \oplus \mathbf{z}^* = y$.)
12. By a similar argument, there is no greatest block.
13. The ordering of the blocks is dense: if $[x]$ is less than $[y]$ (where $x \not\approx y$), then there is a block $[z]$ distinct from both that is between them. Suppose $x \prec y$. As before, $x \oplus y$ is divisible by two (possibly with remainder) so there is a $u \in M$ such that either $x \oplus y = u \oplus u$ or $x \oplus y = u \oplus u \oplus \mathbf{z}^*$. The element u is the average of x and y, and so is between them. Assume $x \oplus y = u \oplus u$ (the other case being similar): if $u \approx x$ then for some standard n:

$$x \oplus y = x \oplus n \oplus x \oplus n,$$

so $y = x \oplus n \oplus n$ and we would have $x \approx y$, against assumption. We conclude that $u \not\approx x$. A similar argument gives $u \not\approx y$.

The non-standard blocks are therefore ordered like the rationals: they form a countable linear ordering without endpoints. It follows that for any two countable non-standard models of true arithmetic, $\mathfrak{M}_1$ and $\mathfrak{M}_2$, their reducts to the language containing $<$ and $=$ only are isomorphic. Indeed, an isomorphism h can be defined as follows: the standard parts of $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are isomorphic to the standard model $\mathfrak{N}$ and hence to each other. The blocks making up the non-standard part are themselves ordered like the rationals and therefore by Theorem 110 are isomorphic; an isomorphism of the blocks can be extended to an isomorphism *within* the blocks by matching up arbitrary elements in each, and then taking the image of the successor of x in $\mathfrak{M}_1$ to be the successor of the image of x in $\mathfrak{M}_2$. Note that it does *not* follow that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are isomorphic in the full language of arithmetic (isomorphism is always relative to a signature), as indeed there are non-isomorphic ways to define addition and multiplication over M_1 and M_2. This also follows from a famous theorem of Vaught's that the number of countable models of a complete theory cannot be 2.)

The interpolation theorem

Our aim in this section is to prove the following:

115. Theorem: (*Craig's Interpolation*) If $\models \varphi \supset \psi$, then there is a sentence θ such that $\models \varphi \supset \theta$ and $\models \theta \supset \psi$, and every constant, function, or predicate symbol (other than $\doteq$) in θ occurs both in φ and ψ. The sentence θ is called an *interpolant* for φ and ψ.

A bit of groundwork is needed before we can proceed with the proof.

116. Definition: A sentence θ *separates* sets of sentences Γ and Δ if and only if $\Gamma \models \theta$ and $\Delta \models \sim\theta$. If no such sentence exists, then Γ and Δ are *inseparable*. The inclusion relations between the classes of models of Γ, Δ and θ are represented below:

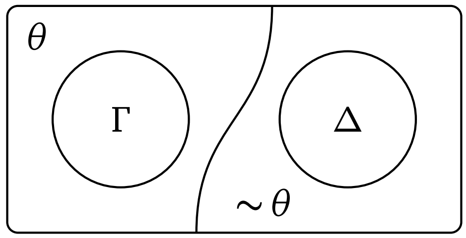

Figure: θ separates Γ and Δ

117. Lemma: Suppose $\mathscr{L}_0$ is the language containing every constant, function and predicate symbol (other than $\doteq$) that occurs in *both* Γ and Δ, and let $\mathscr{L}_0'$ be obtained by the addition of infinitely many new constants c_n for $n \geq 0$. Then if Γ and Δ are inseparable in $\mathscr{L}_0$ then they are also inseparable in $\mathscr{L}_0'$.

Proof. Suppose for contradiction that Γ and Δ are separated in $\mathscr{L}_0'$, so that $\Gamma \models \theta[c/x]$ and $\Delta \models \sim\theta[c/x]$ for some $\theta \in \mathscr{L}_0$ (where c is a new constant — the case where θ contains more than one such new constant is similar). By compactness, there are *finite* subsets Γ_0 of Γ and Δ_0 of Δ such that $\Gamma_0 \models \theta[c/x]$ and $\Delta_0 \models \sim\theta[c/x]$. Let γ be the conjunction of all formulas in Γ_0 and δ the conjunction of all formulas in Δ_0. Then

$$\gamma \models \theta[c/x], \qquad \delta \models \sim\theta[c/x].$$

From the former, by Generalization, we have $\gamma \models \forall x\theta$, and from the latter by Contraposition, $\theta[c/x] \models \sim\delta$, whence also $\forall x\theta \models \sim\delta$. Contraposition again gives $\delta \models \sim\forall x\theta$. By Monotony,

$$\Gamma \models \forall x\theta, \qquad \Delta \models \sim\forall x\theta,$$

so that $\forall x\theta$ separates Γ and Δ in $\mathscr{L}_0$. □

118. Lemma: Suppose that $\Gamma \cup \{\exists x\sigma\}$ and Δ are inseparable, and c is a new constant not in Γ, Δ, or σ. Then $\Gamma \cup \{\exists x\sigma, \sigma[c/x]\}$ and Δ are also inseparable.

Proof. Suppose for contradiction that θ separates $\Gamma \cup \{\exists x\sigma, \sigma[c/x]\}$ and Δ, while at the same time $\Gamma \cup \{\exists x\sigma\}$ and Δ are inseparable. We distinguish two cases:

(*i*) c does not occur in θ: in this case $\Gamma \cup \{\exists x\sigma, \sim\theta\}$ is satisfiable (otherwise θ separates $\Gamma \cup \{\exists x\sigma\}$ and Δ), and it remains so upon adjunction of $\sigma[c/x]$, so θ does not separate $\Gamma \cup \{\exists x\sigma, \sigma[c/x]\}$ and Δ after all.

(ii) c does occur in θ so that θ has the form $\theta[c/x]$. Then we have that

$$\Gamma \cup \{\exists x\sigma, \sigma[c/x]\} \models \theta[c/x],$$

whence $\Gamma, \exists x\sigma \models \forall x(\sigma \supset \theta)$ by the Deduction Theorem and Generalization, and finally $\Gamma \cup \{\exists x\sigma\} \models \exists x\theta$. On the other hand, $\Delta \models \sim\theta[c/x]$ and hence by Generalization $\Delta \models \sim\exists x\theta$. So $\Gamma \cup \{\exists x\sigma\}$ and Δ are separable, a contradiction. $\square$

Proof of Theorem 115. Let $\mathscr{L}_1$ be the language of φ and $\mathscr{L}_2$ be the language of ψ; put $\mathscr{L}_0 = \mathscr{L}_1 \cap \mathscr{L}_2$. For each $i \in \{0,1,2\}$, let $\mathscr{L}'_i$ be obtained from $\mathscr{L}_i$ by adding the infinitely many new constants $c_0, c_1, c_2, \ldots$; we immediately have the following.

Assume also that both φ and $\sim\psi$ are satisfiable, otherwise $\exists x \sim(x \doteq x)$ is an interpolant in the first case, and $\exists x(x \doteq x)$ is an interpolant in the second case.

In order to prove the contrapositive of Craig's Interpolation, assume that there is no interpolant for φ and ψ. Thus, $\{\varphi\}$ and $\{\sim\psi\}$ are inseparable in $\mathscr{L}_0$.

Next, our goal is to extend the pair $(\{\varphi\}, \{\sim\psi\})$ to a maximally inseparable pair (Γ^*, Δ^*). Let $\varphi_0, \varphi_1, \varphi_2, \ldots$ enumerate the sentences of $\mathscr{L}_1$, and $\psi_0, \psi_1, \psi_2, \ldots$ enumerate the sentences of $\mathscr{L}_2$. We define two increasing sequences of sets of sets of sentences (Γ_n, Δ_n), for $n \geq 0$, as follows. Put $\Gamma_0 = \{\varphi\}$ and $\Delta_0 = \{\sim\psi\}$. Assuming (Γ_n, Δ_n) already defined, define Γ_{n+1} and Δ_{n+1} as follows:

- If $\Gamma_n \cup \{\varphi_n\}$ and Δ_n are inseparable in $\mathscr{L}'_0$, put φ_n in Γ_{n+1}; moreover, if φ_n is an existential formula $\exists x\sigma$ then pick a new constant c not occurring in Γ_n, Δ_n, φ_n or ψ_n, and put $\sigma[c/x]$ in Γ_{n+1}.
- If Γ_{n+1} and $\Delta_n \cup \{\psi_n\}$ are inseparable in $\mathscr{L}'_0$, put ψ_n in Δ_{n+1}; moreover, if ψ_n is an existential formula $\exists x\sigma$, then pick a new constant c not occurring in Γ_{n+1}, Δ_n, φ_n or ψ_n, and put $\sigma[c/x]$ in Δ_{n+1}.

Finally, define:

$$\Gamma^* = \bigcup_{n \geq 0} \Gamma_n, \qquad \Delta^* = \bigcup_{n \geq 0} \Delta_n.$$

By simultaneous induction on n we prove:

(a) Γ_n and Δ_n are inseparable in $\mathscr{L}'_0$;

(b) Γ_{n+1} and Δ_n are inseparable in $\mathscr{L}'_0$.

The basis for (a) is given by Lemma 117. For part (b), we need to distinguish three cases:

1. If $\Gamma_0 \cup \{\varphi_0\}$ and Δ_0 are separable, then $\Gamma_1 = \Gamma_0$ and (b) is just (a);
2. If $\Gamma_1 = \Gamma_0 \cup \{\varphi_0\}$, then Γ_1 and Δ_0 are inseparable by construction.
3. It remains to consider the case where φ_0 is existential, so that $\Gamma_1 = \Gamma_0 \cup \{\exists x\sigma, \sigma[c/x]\}$. By construction, $\Gamma_0 \cup \{\exists x\sigma\}$ and Δ_0 are inseparable, so that by Lemma 118 also $\Gamma_0 \cup \{\exists x\sigma, \sigma[c/x]\}$ and Δ_0 are inseparable.

This completes the basis of the induction for (a) and (b) above. Now for the inductive step. For (a), if

$\Delta_{n+1} = \Delta_n \cup \{\psi_n\}$ then Γ_{n+1} and Δ_{n+1} are inseparable by construction (even when ψ_n is existential, by Lemma 118); if $\Delta_{n+1} = \Delta_n$ (because Γ_{n+1} and $\Delta_n \cup \{\psi_n\}$ are separable), then we use the inductive hypothesis on (b). For the inductive step for (b), if $\Gamma_{n+2} = \Gamma_{n+1} \cup \{\varphi_{n+1}\}$ then Γ_{n+2} and Δ_{n+1} are inseparable by construction (even when φ_{n+1} is existential, by Lemma 118); and if $\Gamma_{n+2} = \Gamma_{n+1}$ then we use the inductives case for (a) just proved. This concludes the induction on (a) and (b).

It follows that Γ^* and Δ^* are inseparable; if not, by compactness, there is $n \geq 0$ that separates Γ_n and Δ_n, against (a). In particular, Γ^* and Δ^* are consistent: for if the former or the latter is inconsistent, then they are separated by $\exists x \sim(x \doteq x)$ or $\forall x(x \doteq x)$, respectively.

We now show that Γ^* is maximally consistent in $\mathscr{L}_1'$ and likewise Δ^* in $\mathscr{L}_2'$. For the former, suppose that $\varphi_n \notin \Gamma^*$ and $\sim\varphi_n \notin \Gamma^*$, for some $n \geq 0$. If $\varphi_n \notin \Gamma^*$ then $\Gamma_n \cup \{\varphi_n\}$ is separable from Δ_n, and so there is $\theta \in \mathscr{L}_0'$ such that both:

$$\Gamma^* \models \varphi_n \supset \theta, \qquad\qquad \Delta^* \models \sim\theta.$$

Likewise, if $\sim\varphi_n \notin \Gamma^*$, there is $\theta' \in \mathscr{L}_0'$ such that both:

$$\Gamma^* \models \sim\varphi_n \supset \theta', \qquad\qquad \Delta^* \models \sim\theta'.$$

By propositional logic, $\Gamma^* \models \theta \lor \theta'$ and $\Delta^* \models \sim(\theta \lor \theta')$, so $\theta \lor \theta'$ separates $\Gamma^*$and Δ^*. A similar argument establishes that Δ^* is maximal.

Finally, we show that $\Gamma^* \cap \Delta^*$ is maximally consistent in $\mathscr{L}_0'$. It is obviously consistent, since it is the intersection of consistent sets. To show maximality, let $\sigma \in \mathscr{L}_0'$. Now, Γ^* is maximal in $\mathscr{L}_1' \supseteq \mathscr{L}_0'$, and similarly Δ^* is maximal in $\mathscr{L}_2' \supseteq \mathscr{L}_0'$. It follows that either $\sigma \in \Gamma^*$ or $\sim\sigma \in \Gamma^*$, and either $\sigma \in \Delta^*$ or $\sim\sigma \in \Delta^*$. If $\sigma \in \Gamma^*$ and $\sim\sigma \in \Delta^*$ then σ would separate Γ^* and Δ^*; and if $\sim\sigma \in \Gamma^*$ and $\sigma \in \Delta^*$ then Γ^* and Δ^* would be separated by $\sim\sigma$. Hence, either $\sigma \in \Gamma^* \cap \Delta^*$ or $\sim\sigma \in \Gamma^* \cap \Delta^*$, and $\Gamma^* \cap \Delta^*$ is maximal.

Since Γ^* is maximally consistent, there is a model $\mathfrak{M}_1'$ whose universe M_1 comprises all and only the elements $c^{\mathfrak{M}_1'}$ interpreting the constants — just like in the proof of Theorem 80. Similarly Δ^* has a model $\mathfrak{M}_2'$ whose universe M_2 is given by the interpretations $c^{\mathfrak{M}_2'}$ of the constants.

Let $\mathfrak{M}_1$ be obtained from $\mathfrak{M}_1'$ by dropping interpretations for constants, functions, and predicate symbols in $\mathscr{L}_1' \setminus \mathscr{L}_0'$, and similarly for $\mathfrak{M}_2$. Then the map $\mathsf{h} : M_1 \to M_2$ defined by $\mathsf{h}(c^{\mathfrak{M}_1'}) = c^{\mathfrak{M}_2'}$ is an isomorphism in $\mathscr{L}_0'$, because $\Gamma^* \cap \Delta^*$ is maximally consistent in $\mathscr{L}_0'$, as shown. This follows because any $\mathscr{L}_0'$-sentence either belongs to both Γ^* and Δ^*, or to neither: so $c^{\mathfrak{M}_1'} \in P^{\mathfrak{M}_1'}$ if and only if $Pc \in \Gamma^*$ if and only if $Pc \in \Delta^*$ if and only if $c^{\mathfrak{M}_2'} \in P^{\mathfrak{M}_2'}$. The other conditions satisfied by isomorphisms can be established similarly.

Let us now define a model $\mathfrak{M}$ for the language $\mathscr{L}_1 \cup \mathscr{L}_2$ as follows:

- The universe M is just M_2 i.e., the set of all elements $c^{\mathfrak{M}_2'}$;
- If a predicate P is in $\mathscr{L}_2 \setminus \mathscr{L}_1$ then $P^{\mathfrak{M}} = P^{\mathfrak{M}_2'}$;

- ► If a predicate P is in $\mathscr{L}_1 \setminus \mathscr{L}_2$ then $P^{\mathfrak{M}} = \mathsf{h}(P^{\mathfrak{M}'_2})$, i.e., $\langle c_1^{\mathfrak{M}'_2}, \ldots, c_n^{\mathfrak{M}'_2} \rangle \in P^{\mathfrak{M}}$ if and only if $\langle c_1^{\mathfrak{M}'_1}, \ldots, c_n^{\mathfrak{M}'_1} \rangle \in P^{\mathfrak{M}'_1}$.
- ► If a predicate P is in $\mathscr{L}_0$ then $P^{\mathfrak{M}} = P^{\mathfrak{M}'_2} = \mathsf{h}(P^{\mathfrak{M}'_1})$.
- ► Function symbols of $\mathscr{L}_1 \cup \mathscr{L}_2$, including constants, are handled similarly.

Finally, one shows by induction on formulas that $\mathfrak{M}$ agrees with $\mathfrak{M}'_1$ on all formulas of $\mathscr{L}'_1$ and with $\mathfrak{M}'_2$ on all formulas of $\mathscr{L}'_2$. In particular, $\mathfrak{M} \models \Gamma^* \cup \Delta^*$, whence $\mathfrak{M} \models \varphi$ and $\mathfrak{M} \models \sim\psi$, and $\not\models \varphi \supset \psi$. This concludes the proof of Craig's Interpolation Theorem. □

We conclude this section with a simple consequence of the Interpolation Theorem concerning definability.

119. Definition: Given a language $\mathscr{L}$ and predicates P and P' not in $\mathscr{L}$, a set $\Sigma(P)$ of sentences of $\mathscr{L} \cup \{P\}$ *implicitly defines P* if and only if

$$\Sigma(P) \cup \Sigma(P') \models \forall x_1 \ldots \forall x_n (P x_1 \ldots x_n \leftrightarrow P' x_1 \ldots x_n),$$

where $\Sigma(P')$ is the result of uniformly replacing P' for P in $\Sigma(P)$.

In other words, for any model $\mathfrak{A}$ and $R, R' \subseteq A^n$, if both $(\mathfrak{A}, R) \models \Sigma(P)$ and $(\mathfrak{A}, R') \models \Sigma(P')$, then $R = R'$; where $(\mathfrak{A}, R)$ is the expansion of $\mathscr{L}$ to $\mathscr{L} \cup \{P\}$ such that $P^{\mathfrak{A}} = R$, and similarly for $(\mathfrak{A}, R')$.

120. Definition: Given a language $\mathscr{L}$ and a predicate P not in $\mathscr{L}$, a set $\Sigma(P)$ of sentences of $\mathscr{L} \cup \{P\}$ *explicitly defines P* if and only if there is a formula $\theta(x_1, \ldots, x_n)$ of $\mathscr{L}$ such that

$$\Sigma(P) \models \forall x_1 \ldots x_n (P x_1 \ldots x_n \leftrightarrow \theta(x_1, \ldots, x_n)).$$

121. Theorem: (*Beth Definability Theorem*) A set $\Sigma(P)$ of $\mathscr{L} \cup \{P\}$-formulas implicitly defines P if and only $\Sigma(P)$ explicitly defines P.

Proof. If $\Sigma(P)$ explicitly defines P then both

$$\Sigma(P) \models \forall x_1 \ldots x_n (P x_1 \ldots x_n \leftrightarrow \theta(x_1, \ldots, x_n))$$
$$\Sigma(P') \models \forall x_1 \ldots x_n (P' x_1 \ldots x_n \leftrightarrow \theta(x_1, \ldots, x_n))$$

and the conclusion follows. For the converse: assume that $\Sigma(P)$ implicitly defines P, and add countably many new constants to $\mathscr{L}$. Then

$$\Sigma(P) \cup \Sigma(P') \models P c_1 \ldots c_n \to P' c_1 \ldots c_n.$$

By compactness, there are finite sets $\Delta_0 \subseteq \Sigma(P)$ and $\Delta_1 \subseteq \Sigma(P')$ such that

$$\Delta_0 \cup \Delta_1 \models P c_1 \ldots c_n \to P' c_1 \ldots c_n.$$

Let $\delta(P)$ be the conjunction of all sentences $\varphi(P)$ such that either $\varphi(P) \in \Delta_0$ or $\varphi(P') \in \Delta_1$ and let $\delta(P')$ be the conjunction of all sentences $\varphi(P')$ such that either $\varphi(P) \in \Delta_0$ or $\varphi(P') \in \Delta_1$. Then $\delta(P) \wedge \delta(P') \models Pc_1 \ldots c_n \rightarrow P'c_1 \ldots c_n$. Re-arranging such that each predicate occurs on one side of $\models$:

$$\delta(P) \wedge Pc_1 \ldots c_n \models \delta(P') \rightarrow P'c_1 \ldots c_n.$$

By Craig's Interpolation there is a sentence $\theta(c_1, \ldots, c_n)$ not containing P or P' such that:

$$\delta(P) \wedge Pc_1 \ldots c_n \models \theta(c_1, \ldots, c_n); \qquad \theta(c_1, \ldots, c_n) \models \delta(P') \rightarrow P'c_1 \ldots c_n.$$

From the former of these two entailments we have: $\delta(P) \models Pc_1 \ldots c_n \rightarrow \theta(c_1, \ldots, c_n)$. And from the latter, since an $\mathscr{L} \cup \{P\}$-model $(\mathfrak{A}, R) \models \varphi(P)$ if and only if the corresponding $\mathscr{L} \cup \{P'\}$-model $(\mathfrak{A}, R) \models \varphi(P')$, we have $\theta(c_1, \ldots, c_n) \models \delta(P) \rightarrow Pc_1 \ldots c_n$, from which:

$$\delta(P) \models \theta(c_1, \ldots, c_n) \rightarrow Pc_1 \ldots c_n.$$

Putting the two together, $\delta(P) \models Pc_1 \ldots c_n \leftrightarrow \theta(c_1, \ldots, c_n)$, and by monotony and generalization also

$$\Sigma(P) \models \forall x_1, \ldots \forall x_n (Px_1 \ldots x_n \leftrightarrow \theta(x_1, \ldots, x_n)). \qquad \square$$

Lindström's Theorem

In this section we aim to prove Lindström's characterization of first-order logic as the maximal logic for which (given certain further constraints) the Compactness and the Downward Löwenheim-Skolem theorems hold (corollary 82 and proposition 88). First, we need a more general characterization of the general class of logics to which the theorem applies. For the purposes of this section, we restrict ourselves to *relational* languages, i.e., languages whose signatures contain predicate symbols and individual constants, but no function symbols.

122. Definition: An *abstract logic* is a pair $(L, \models_L)$, where L is a function that assigns to each signature $\mathfrak{s}$ a set $L(\mathfrak{s})$ of sentences, and $\models_L$ is a relation between structures for the signature $\mathfrak{s}$ and members of $L(\mathfrak{s})$.

Notice that we are still employing the same notion of structure for a given signature as for first-order logic, but we do not presuppose that sentences are build up from the basic symbols in $\mathfrak{s}$ in the usual way, nor that the relation $\models_L$ is recursively defined in the same way as for first-order logic. So for instance the definition is intended to capture the case where sentences in $(L, \models_L)$ contain infinitely long conjunctions or disjunction, or quantifiers other than $\exists$ and $\forall$ (e.g., "there are infinitely many x such that ..."), or perhaps infinitely long quantifier prefixes. To emphasize that "sentences" in $L(\mathfrak{s})$ need not be ordinary sentences of first-order logic, we use variables $\alpha, \beta, \ldots$ to range over them.

123. Definition: Let $\mathsf{Mod}_L(\alpha)$ denote the class $\{\mathfrak{M} : \mathfrak{M} \models_L \alpha\}$. If the signature needs to be made explicit, we write $\mathsf{Mod}^{\mathfrak{s}}_L(\alpha)$. Two structures $\mathfrak{M}$ and $\mathfrak{N}$ for $\mathfrak{s}$ are *elementarily equivalent in* $(L, \models_L)$, written $\mathfrak{M} \equiv_L \mathfrak{N}$, if the same sentences from $L(\mathfrak{s})$ are true in each.

124. Definition: An abstract logic $(L, \models_L)$ for the signature $\mathfrak{s}$ is *normal* if it satisfies the following properties:

1. (*L-Monotony*) For signatures $\mathfrak{s}$ and $\mathfrak{t}$, if $\mathfrak{s} \subseteq \mathfrak{t}$, then $L(\mathfrak{s}) \subseteq L(\mathfrak{t})$.
2. (*Expansion Property*) For each $\alpha \in L(\mathfrak{s})$ there is a *finite* subset $\mathfrak{s}'$ of $\mathfrak{s}$ such that the relation $\mathfrak{M} \models_L \alpha$ depends only on the reduct of $\mathfrak{M}$ to $\mathfrak{s}'$; i.e., if $\mathfrak{M}$ and $\mathfrak{N}$ have the same reduct to $\mathfrak{s}'$ then $\mathfrak{M} \models_L \alpha$ if and only if $\mathfrak{N} \models_L \alpha$.
3. (*Isomorphism Property*) If $\mathfrak{M} \models_L \alpha$ and $\mathfrak{M} \simeq \mathfrak{N}$ then also $\mathfrak{N} \models_L \alpha$.
4. (*Renaming Property*) The relation $\models_L$ is preserved under renaming: if the signature $\mathfrak{s}'$ is obtained from $\mathfrak{s}$ by replacing each symbol P by a symbol P' of the same arity and each constant c by a distinct constant c', then for each structure $\mathfrak{M}$ and sentence α, $\mathfrak{M} \models_L \alpha$ if and only if $\mathfrak{M}' \models_L \alpha'$, where $\mathfrak{M}'$ is the $\mathfrak{s}'$-structure corresponding to $\mathfrak{s}$ and $\alpha' \in L(\mathfrak{s}')$.
5. (*Boolean Property*) The abstract logic $(L, \models_L)$ is closed under the Boolean connectives in the sense that for each $\alpha \in L(\mathfrak{s})$ there is an $\beta \in L(\mathfrak{s})$ such that $\mathfrak{M} \models_L \beta$ if and only if $\mathfrak{M} \not\models_L \alpha$, and for each α and β there is a γ such that $\mathsf{Mod}_L(\gamma) = \mathsf{Mod}_L(\alpha) \cap \mathsf{Mod}_L(\beta)$. Similarly for atomic formulas and the other connectives.
6. (*Quantifier Property*) For each constant c in $\mathfrak{s}$ and $\alpha \in L(\mathfrak{s})$ there is a $\beta \in L(\mathfrak{s})$ such that
$$\mathsf{Mod}^{\mathfrak{s}'}_L(\beta) = \{\mathfrak{M} : (\mathfrak{M}, a) \in \mathsf{Mod}^{\mathfrak{s}}_L(\alpha) \text{ for some } a \in M\},$$
where $\mathfrak{s}' = \mathfrak{s} - \{c\}$ and $(\mathfrak{M}, a)$ is the expansion of $\mathfrak{M}$ to $\mathfrak{s}$ assigning a to c.
7. (*Relativization Property*) Given a sentence $\alpha \in L(\mathfrak{s})$ and symbols $R, c_1, \ldots, c_n$ not in $\mathfrak{s}$, there is a sentence $\beta \in L(\mathfrak{s} \cup \{R, c_1, \ldots, c_n\})$ called the *relativization* of α to $Rxc_1 \ldots c_n$, such that for each structure $\mathfrak{M}$:
$$(\mathfrak{M}, X, b_1, \ldots, b_n) \models_L \beta) \text{ if and only if } \mathfrak{N} \models_L \alpha,$$
where $\mathfrak{N}$ is the substructure of $\mathfrak{M}$ with universe $N = \{a \in M : R^{\mathfrak{M}}(a, b_1, \ldots, b_n)\}$ (see Remark 86), and $(\mathfrak{M}, X, b_1, \ldots, b_n)$ is the expansion of $\mathfrak{M}$ interpreting $R, c_1, \ldots, c_n$ by $X, b_1, \ldots, b_n$, respectively (with $X \subseteq M^{n+1}$).

125. Definition: Given two abstract logics $(L_1, \models_{L_1})$ and $(L_2, \models_{L_2})$ we say that the latter is at least as expressive as the former, written $(L_1, \models_{L_1}) \leq (L_2, \models_{L_2})$, if for each signature $\mathfrak{s}$ and sentence $\alpha \in L_1(\mathfrak{s})$ there is a sentence $\beta \in L_2(\mathfrak{s})$ such that $\mathsf{Mod}^{\mathfrak{s}}_{L_1}(\alpha) = \mathsf{Mod}^{\mathfrak{s}}_{L_2}(\beta)$. The logics $(L_1, \models_{L_1})$ and $(L_2, \models_{L_2})$ are *equivalent* if $(L_1, \models_{L_1}) \leq (L_2, \models_{L_2})$ and $(L_2, \models_{L_2}) \leq (L_1, \models_{L_1})$.

126. Remark: First-order logic, i.e., the abstract logic $(F, \models)$, is normal. In fact, the above properties are mostly straightforward for first-order logic. We just remark that the expansion property comes down to local determination, and that the relativization of a sentence φ to $Rxc_1 \ldots c_n$ is obtained by replacing each subformula $\forall x \psi$ by $\forall x(Rxc_1 \ldots c_n \to \psi)$. Moreover, if $(L, \models_L)$ is normal, then $(F, \models) \leq (L, \models_L)$, as it can be can shown by induction on formulas. Accordingly, with no loss in generality, we can assume that every first-order sentence belongs to every normal logic.

We now give the obvious extensions of compactness and Löwenheim-Skolem to the case of abstract logics.

127. Definition: Let $(L, \models_L)$ be an abstract logic; then:

- ► $(L, \models_L)$ has the Compactness Property if each set Γ of $L(\mathfrak{s})$-sentences is satisfiable whenever each finite $\Gamma_0 \subseteq \Gamma$ is satisfiable.
- ► $(L, \models_L)$ has the Downward Löwenheim-Skolem property if any satisfiable Γ has a model of cardinality at most countably infinite.

The notion of partial isomorphism from Definition 99 is purely "algebraic" (i.e., given without reference to the sentences of the language but only to the signature $\mathfrak{s}$ of the structures), and hence it applies to the case of abstract logics. In case of first-order logic, we know from Theorem 101 that if two structures are partially isomorphic then they are elementarily equivalent. That proof does not carry over to abstract logics, for induction on formulas need not be available for arbitrary $\alpha \in L(\mathfrak{s})$, but the theorem is true nonetheless, provided the Löwenheim-Skolem property holds.

128. Theorem: Suppose $(L, \models_L)$ is a normal logic with the Löwenheim-Skolem property. Then any two structures that are partially isomorphic are elementarily equivalent in $(L, \models_L)$.

Proof. Suppose $\mathfrak{M} \simeq_p \mathfrak{N}$, but for some α also $\mathfrak{M} \models_L \alpha$ while $\mathfrak{N} \not\models_L \alpha$. By the Isomorphism Property we can assume that M and N are disjoint, and by the expansion property we can assume that $\alpha \in L(\mathfrak{s})$ for a finite signature $\mathfrak{s}$. Let $\mathscr{I}$ be a set of partial isomorphisms, and with no loss in generality we can also assume that if $p \in \mathscr{I}$ and $q \subseteq p$ then also $q \in \mathscr{I}$.

Where $M^{<\omega}$ is the set of finite sequences over M: let S be the ternary relation over $M^{<\omega}$ representing concatenation, i.e., if $\mathbf{a}, \mathbf{b}, \mathbf{c} \in M^{<\omega}$ then $S(\mathbf{a}, \mathbf{b}, \mathbf{c})$ holds if and only if $\mathbf{c}$ is the concatenation of $\mathbf{a}$ and $\mathbf{b}$; and let T be the ternary relation such that $T(\mathbf{a}, b, \mathbf{c})$ holds for $b \in M$ and $\mathbf{a}, \mathbf{c} \in M^{<\omega}$ if and only if $\mathbf{a} = a_1, \ldots a_n$ and $\mathbf{c} = a_1, \ldots a_n, b$. Pick new 3-place predicate symbols P and Q and form the structure $\mathfrak{M}^*$ having the universe $M \cup M^{<\omega}$, having $\mathfrak{M}$ as a substructure, and interpreting P and Q by the concatenation relations S and T (so $\mathfrak{M}^*$ is in the signature $\mathfrak{s} \cup \{P, Q\}$).

Define $N^{<\omega}$, S', T', P', Q' and $\mathfrak{N}^*$ analogously. Since by hypothesis $\mathfrak{M} \simeq_p \mathfrak{N}$, there is a relation I between $M^{<\omega}$ and $N^{<\omega}$ such that $I(\mathbf{a}, \mathbf{b})$ holds if and only if $\mathbf{a}$ and $\mathbf{b}$ are isomorphic and satisfying the back-and-forth condition of Definition 99. Now, let $\mathfrak{A}$ be the structure whose universe is the union of the universes of $\mathfrak{M}^*$ and $\mathfrak{N}^*$, having $\mathfrak{M}^*$ and $\mathfrak{N}^*$ as substructures, in the signature with one extra binary symbol R interpreted by the relation I and predicates denoting the universes M^* and N^*.

The crucial observation is that in the language of the structure $\mathfrak{A}$ there is a first-order sentence θ_1 true in $\mathfrak{A}$ saying that $\mathfrak{M} \models_L \alpha$ and $\mathfrak{N} \not\models_L \alpha$ (this requires the Relativization Property), as well as a *first-order* sentence θ_2 true in $\mathfrak{A}$ saying that $\mathfrak{M} \simeq_p \mathfrak{N}$ via the partial isomorphism I. By the Löwenheim-Skolem Property, θ_1 and θ_2 are jointly true in a countable model $\mathfrak{A}_0$ containing partially isomorphic substruc-

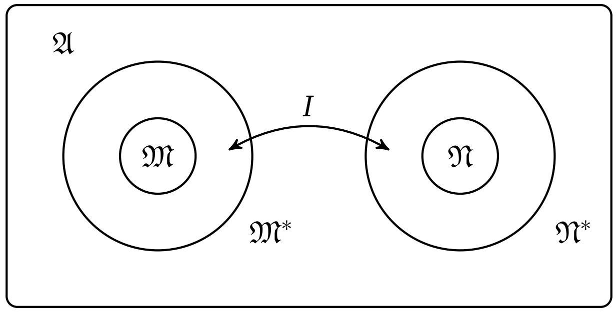

Figure: the structure $\mathfrak{A}$ with the internal partial isomorphism.

tures $\mathfrak{M}_0$ and $\mathfrak{N}_0$ such that $\mathfrak{M}_0 \models_L \alpha$ and $\mathfrak{N}_0 \not\models_L \alpha$. But countable partially isomorphic structures are in fact isomorphic by Theorem 100, contradicting the Isomorphism Property of normal logics. □

129. Lemma: Suppose $\alpha \in L(\mathfrak{s})$, with $\mathfrak{s}$ finite, and assume also that there is $n \in \mathbb{N}$ such that for any two structures $\mathfrak{M}$ and $\mathfrak{N}$, if $\mathfrak{M} \equiv_n \mathfrak{N}$ and $\mathfrak{M} \models_L \alpha$ then also $\mathfrak{N} \models_L \alpha$. Then α is equivalent to a first-order sentence, i.e., there is a first-order θ such that $\mathsf{Mod}_L(\alpha) = \mathsf{Mod}_L(\theta)$.

Proof. Let n be such that any two n-equivalent structures $\mathfrak{M}$ and $\mathfrak{N}$ agree on the value assigned to α. Recall that there are only finitely many first-order sentences in a finite signature that have quantifier rank no greater than n, up to logical equivalence. Now, for each fixed structure $\mathfrak{M}$ let $\theta_{\mathfrak{M}}$ be the conjunction of all first-order sentences φ true in $\mathfrak{M}$ with $\mathsf{qr}(\varphi) \leq n$ (this conjunction is finite), so that $\mathfrak{N} \models \theta_{\mathfrak{M}}$ if and only if $\mathfrak{N} \equiv_n \mathfrak{M}$. Then put $\theta = \bigvee\{\theta_{\mathfrak{M}} : \mathfrak{M} \models_L \alpha\}$; this disjunction is also finite (up to logical equivalence).

The conclusion $\mathsf{Mod}_L(\alpha) = \mathsf{Mod}_L(\theta)$ follows. In fact, if $\mathfrak{N} \models_L \theta$ then for some $\mathfrak{M} \models_L \alpha$ we have $\mathfrak{N} \models \theta_{\mathfrak{M}}$, whence also $\mathfrak{N} \models_L \alpha$ (by the hypothesis of the lemma). Conversely, if $\mathfrak{N} \models_L \alpha$ then $\theta_{\mathfrak{N}}$ is a disjunct in θ, and since $\mathfrak{N} \models \theta_{\mathfrak{N}}$, also $\mathfrak{N} \models_L \theta$. □

130. Lindström's Theorem: Suppose $(L, \models_L)$ has the compactness and the Löwenheim-Skolem Properties. Then $(L, \models_L) \leq (F, \models)$ (where the latter is first-order logic).

Proof. By Lemma 129, suffices to show that for any $\alpha \in L(\mathfrak{s})$, with $\mathfrak{s}$ finite, there is $n \in \mathbb{N}$ such that for any two structures $\mathfrak{M}$ and $\mathfrak{N}$: if $\mathfrak{M} \equiv_n \mathfrak{N}$ then $\mathfrak{M}$ and $\mathfrak{N}$ agree on α. For then α is equivalent to a first order sentence, from which $(L, \models_L) \leq (F, \models)$ follows. Since we are working in a finite, purely relational signature, by Theorem 107 we can replace the statement that $\mathfrak{M} \equiv_n \mathfrak{N}$ by the corresponding algebraic statement that $I_n(\varnothing, \varnothing)$.

Given α, suppose towards a contradiction that for each n there are structures $\mathfrak{M}_n$ and $\mathfrak{N}_n$ such that $I_n(\varnothing, \varnothing)$, but (say) $\mathfrak{M}_n \models_L \alpha$ whereas $\mathfrak{M}_n \not\models_L \alpha$. By the Isomorphism Property we can assume that all the $\mathfrak{M}_n$s interpret the constants of the language by the same objects; furthermore, since there are only finitely many atomic sentences in the language, we may also assume that they satisfy the same atomic sentences (we can take a subsequence of the $\mathfrak{M}$'s otherwise). Let $\mathfrak{M}$ be the union of all the $\mathfrak{M}_n$s, i.e., the unique minimal structure having each $\mathfrak{M}_n$ as a substructure. As in the proof of 128, let $\mathfrak{M}^*$ be the extension of $\mathfrak{M}$ with universe $M \cup M^{<\omega}$, in the expanded language comprising concatenation predicates P and Q.

Similarly, define $\mathfrak{N}_n$, $\mathfrak{N}$ and $\mathfrak{N}^*$. Now let $\mathfrak{A}$ be the structure whose universe comprises the universes of $\mathfrak{M}^*$ and $\mathfrak{N}^*$ as well as the natural number $\mathbb{N}$ along with their natural ordering $\leq$, in the signature with extra predicates representing the universes M, N, $M^{<\omega}$ and $N^{<\omega}$ as well as predicates coding the domains of $\mathfrak{M}_n$ and $\mathfrak{N}_n$ in the sense that:

$$M_n = \{a \in M : R(a,n)\}; \qquad N_n = \{a \in N : S(a,n)\};$$
$$M_n^{<\omega} = \{a \in M^{<\omega} : R(a,n)\}; \qquad N_n^{<\omega} = \{a \in N^{<\omega} : S(a,n)\}.$$

The structure $\mathfrak{A}$ also has a ternary relation J such that $J(n,\mathbf{a},\mathbf{b})$ holds if and only if $I_n(\mathbf{a},\mathbf{b})$.

Now there is a sentence θ is the language $\mathfrak{s}$ augmented by R, S, J, etc. saying that $\leq$ is a discrete linear ordering with first but no last element and such that $\mathfrak{M}_n \models \alpha$, $\mathfrak{N}_n \not\models \alpha$, and for each n in the ordering $J(n,\mathbf{a},\mathbf{b})$ holds if and only if $I_n(\mathbf{a},\mathbf{b})$.

Using the Compactness Property, we can find a model $\mathfrak{A}^*$ of θ in which the ordering contains a non-standard element n^*. In particular then $\mathfrak{A}^*$ will contain substructures $\mathfrak{M}_{n^*}$ and $\mathfrak{N}_{n^*}$ such that $\mathfrak{M}_{n^*} \models_L \alpha$ and $\mathfrak{N}_{n^*} \not\models_L \alpha$. But now we can define a set $\mathscr{I}$ of pairs of k-tuples from M_{n^*} and N_{n^*} by putting $(\mathbf{a},\mathbf{b}) \in \mathscr{I}$ if and only if $J(n^*-k,\mathbf{a},\mathbf{b})$, where k is the length of $\mathbf{a}$ and $\mathbf{b}$. Since n^* is non standard, for each standard k we have that $n^*-k > 0$, and the set $\mathscr{I}$ witnesses the fact that $\mathfrak{M}_{n^*} \simeq_p \mathfrak{N}_{n^*}$. But by Theorem 128, $\mathfrak{M}_{n^*}$ is L-equivalent to $\mathfrak{N}_{n^*}$, a contradiction. $\square$

Part V

Problems Sets

Problem Set I: syntactic matters

1. Definition: Let $\mathscr{X}$ be a set of basic symbols comprising: $K, N, p_0, p_1, p_2, \ldots$, which are assumed to be all distinct. We refer to arbitrary strings (n-tuples) over $\mathscr{X}$ as *expressions*, and denote concatenation by juxtaposition. Let WFF, the set of well-formed formulas, be the smallest set of expressions satisfying the clauses:

1. $p_n \in \mathsf{WFF}$ for all $n \geq 0$;
2. if $\varphi, \psi \in \mathsf{WFF}$, then both $K\varphi\psi$ and $N\varphi$ are all in WFF.

The intended interpretation is that the objects p_n are atomic sentences, K and N represent conjunction and negation, respectively, and WFF is the set of all sentences. So the following are in WFF: p_{13}, Np_7, Kp_5p_{123}, NKp_5p_{123}; but the following are not: Kp_5, $p_{123}Kp_5$. Notice that we do not have parentheses; this way of writing sentences in WFF is referred to as "Polish notation." As a consequence of the definition we have the following:

2. Theorem: (*Principle of Induction on* WFF) Let B be any subset of WFF satisfying conditions (1.) and (2.) above (i.e., B contains $\{p_n : n \geq 0\}$ and is closed under K and N); then $B = \text{WFF}$.

3. Definition: Define a function $\mathbf{r} : \mathscr{X} \to \mathbb{Z}$ ($\mathbb{Z}$ the set of the integers) explicitly by putting $\mathbf{r}(x) = 1 - k$, where k is the number of symbols that have to follow x in order to obtain a WFF. In other words:

$$\mathbf{r}(p_n) = 1; \qquad \mathbf{r}(N) = 0; \qquad \mathbf{r}(K) = -1.$$

We then "lift" the function $\mathbf{r}$ to a function $\bar{\mathbf{r}}$ defined over arbitrary expressions by putting:

$$\bar{\mathbf{r}}(x_1 \dots x_k) = \mathbf{r}(x_1) + \dots + \mathbf{r}(x_k).$$

By the properties of addition, if φ is the concatenation of expressions $\psi_1 \dots, \psi_m$, then $\bar{\mathbf{r}}(\varphi) = \bar{\mathbf{r}}(\psi_1) + \dots + \bar{\mathbf{r}}(\psi_m)$ (you may freely help yourself to this fact in the problems below).

4. Problem: Show that if $\varphi \in \text{WFF}$ then $\bar{\mathbf{r}}(\varphi) = 1$. *Hint*: use induction on WFFs.

5. Problem: Prove that any terminal (not necessarily proper) segment of a WFF is a concatenation of one or more WFFs. (A terminal segment of $x_1 \dots x_n$ is any k-tuple $x_k x_{k+1} \dots x_n$ for $1 \leq k \leq n$.) *Hint*: use induction on WFFs.

6. Problem: Prove that no proper initial segment of a WFF is a WFF.
Hint: suppose $\varphi = \psi_1 \psi_2$ and consider $\bar{\mathbf{r}}(\varphi)$. Use the previous two problems.

7. Problem: We know from problem 4 that if $\varphi \in \text{WFF}$, then $\bar{\mathbf{r}}(\varphi) = 1$. Prove that the converse is not true by exhibiting an expression ψ for which $\bar{\mathbf{r}}(\psi) = 1$ but ψ is not in WFF.

8. Problem: Let φ be any expression, not necessarily a WFF, and suppose that for every terminal segment ψ of φ it holds that $\bar{\mathbf{r}}(\psi) > 0$. Prove that φ is a concatenation of $\bar{\mathbf{r}}(\varphi)$ WFFs.
Hint: proceed by complete mathematical induction (not on WFF, but) on the number of symbols in φ. For the inductive step, consider $\varphi = x_1 \dots x_n$, with $n > 1$. Then for each k (where $1 < k \leq n$), $x_k \dots x_n$ is a concatenation of $\bar{\mathbf{r}}(x_k \dots x_n)$ WFF's (why?). Take up in turn the different cases, according as x_1 is p_i, K, or N. For instance, consider the case where x_1 is N: we also know that $\bar{\mathbf{r}}(x_2, \dots, x_n) > 0$ (why?), so $x_2, \dots, x_n$ is a concatenation of > 0 WFFs, and can be written $\psi x_k, \dots, x_n$ for some WFF ψ. Compute $\bar{\mathbf{r}}(x_1, \dots, x_n)$.

9. Problem: Let φ be any expression, not necessarily a WFF. Prove that φ is a WFF if and only if both the following hold: $\bar{\mathbf{r}}(\varphi) = 1$ and for every terminal segment ψ of φ, $\bar{\mathbf{r}}(\psi) > 0$. *Hint*: use problems (5), (8).

10. Problem: Show that if $\varphi \in \text{WFF}$, and ψ is a *proper* initial segment of φ, then $\bar{\mathbf{r}}(\psi) < 1$. *Hint*: use problem (5).

11. Problem: Prove the unique readability theorem for WFF. In other words, show that K and N

(when viewed as operations on WFF) are one-one and have ranges disjoint with each other and with $\{p_i : i \geq 0\}$.

Problem Set II: semantic matters

1. Problem: Let φ be a formula all of whose atomic subformulæ are among $p_1, \ldots, p_k$; also, let v_1 and v_2 be valuations such that $v_1(p_i) = \overline{v}_2(\theta_i)$ for each $i \in \{1, \ldots, k\}$. Show that

$$\overline{v}_1(\varphi) = \overline{v}_2(\varphi[{}^{\theta_1}/_{p_1}, \ldots, {}^{\theta_k}/_{p_k}]).$$

2. Definition: A k-place *truth function* (also called a *Boolean function*) is a function $f : \{\mathsf{t}, \mathsf{f}\}^k \to \{\mathsf{t}, \mathsf{f}\}$. A formula φ whose atomic components are $p_1, \ldots, p_k$ *generates* (or *realizes*) a truth function f if and only if $\overline{v}(\varphi) = f(v(p_1), \ldots, v(p_k))$ for every valuation v.

3. Definition: A set of connectives is *(truth-functionally) complete* if and only if for every k, every k-place truth function is realized by a formula using only connectives in the set.

4. Problem: Show that the set of connectives $\{\supset, \sim\}$ is complete. *Hint*: we need to show that for every k, every k-place truth function is realized by a formula using only connectives in the set. Here is the case for $k = 2$: given a 2-place truth function f, construct a table:

t	t	$f(\mathsf{t},\mathsf{t})$
t	f	$f(\mathsf{t},\mathsf{f})$
f	t	$f(\mathsf{f},\mathsf{t})$
f	f	$f(\mathsf{f},\mathsf{f})$

For each row where $f(\mathbf{x}, \mathbf{y}) = \mathsf{t}$ write a sentence using atomic letters p_1 and p_2 representing the combination of truth values on that row. Let φ be the disjunction of all formula thus obtained. Show that φ realizes f.

5. Definition: Let # be the 3-place connective returning value t when and only when a majority of its arguments returns value t, giving rise to the following truth-table:

x	y	z	$\#(x,y,z)$
t	t	t	t
t	t	f	t
t	f	t	t
t	f	f	f
f	t	t	t
f	t	f	f
f	f	t	f
f	f	f	f

6. Problem: Show that any formula $\varphi(p,q)$ using the two propositional variables p and q and the connective # (but no other variables or connectives) is either equivalent to p or to q.

7. Problem: Show that any formula $\varphi(p,q)$ using the two propositional variables p and q and the two connectives # and $\sim$ (but no other variables or connectives) is equivalent either to one of the formulas p, q, $\sim p$, or to $\sim q$.

8. Problem: Consider the set of connectives $\{\#, \sim\}$: either show that this set is truth-functionally complete (perhaps using the techniques from Problem 4), or prove that it is not.

9. Problem: Suppose that $\models \varphi \supset \psi$ and that φ is not a contradiction and ψ not a tautology. Show that there is a propositional variable p that occurs in both φ and ψ.

10. Definition: If $\models \varphi \supset \psi$, then an *interpolant* for φ and ψ is a formula θ such that:

(*i*) $\models \varphi \supset \theta$ and $\models \theta \supset \psi$;

(*ii*) every propositional variable p in θ occurs in both φ and ψ.

11. Problem: Show that if $\models \varphi \supset \psi$, and φ is neither a contradiction nor ψ a tautology, then there is an interpolant for φ and ψ.

Hint: Suppose $\models \varphi \supset \psi$; by the previous problem φ and ψ share at least one propositional variable. For simplicity, let us write $\varphi(p,q)$ and $\psi(q,r)$ where φ and ψ contain just the variables shown. Define $\bot = \sim(q \supset q)$ and $\top = q \supset q$. Consider the formula $\varphi(\top,q) \vee \varphi(\bot,q)$, where $\varphi(\top,q)$ is $\varphi[{}^{\top}/{}_{p}]$, and similarly for $\varphi(\bot,q)$.

The formula $\varphi(\top,q) \vee \varphi(\bot,q)$ is desired interpolant. For if $\overline{v}(\varphi) = \mathsf{t}$, then either $v(p) = \mathsf{t}$ or $v(p) = \mathsf{f}$; if the former then $\overline{v}(\varphi(\top,q)) = \mathsf{t}$ (why?), and if the latter $\overline{v}(\varphi(\bot,q)) = \mathsf{t}$ (why?). So $\varphi \models \varphi(\top,q) \vee \varphi(\bot,q)$.

Conversely, to show $\varphi(\top,q) \vee \varphi(\bot,q) \models \psi$, suppose $\overline{v}(\varphi(\top,q)) = \mathsf{t}$; let v' be just like v except that $v'(p) = \mathsf{t}$. Then $\overline{v'}(\varphi(p,q)) = \mathsf{t}$ (why?), and since by hypothesis $\overline{v'}(\psi) = \mathsf{t}$, also $\overline{v}(\psi) = \mathsf{t}$ (why?). Thus, $\varphi(\top,q) \models \psi$, and similarly $\varphi(\bot,q) \models \psi$.

12. Problem: Suppose that $\models \varphi(p,q) \supset \psi(q,r)$, where neither the former is a contradiction nor the latter a tautology. Show that $\psi(q,\top) \wedge \psi(q,\bot)$ is an interpolant for φ and ψ.

If φ is a contradiction or ψ is a tautology, then an interpolant can be found if we assume that $\top$ and $\bot$ are 0-place connectives (propositional constants).

13. Problem: Show that propositional logic is *Post-complete* in the following sense: if φ is any formula such that $\not\vdash \varphi$ then the system obtained by adding all the substitution instances of φ as further axioms (in addition to our three axiom schemas) is inconsistent.

Hint: let $\vdash^*$ be the system comprising all the old axioms as well as all the substitution instances of φ. Clearly $\vdash^*$ extends $\vdash$. Since $\not\vdash \varphi$ there is a valuation v^* s.t. $\overline{v^*}(\varphi) = \mathsf{f}$ (why?). Let $p_1 \dots p_k$ be all the atomic components of φ. Show that there is a substitution instance φ^* of φ such that $\models \sim \varphi^*$, and use completeness — again! — to reach the desired conclusion.

Problem Set III: applications of compactness

1. Definition: A map is k-colorable if and only if it is possible to assign to each region one of k distinct colors in such a way that no two regions that share a border are assigned the same color. (Regions sharing only one point don't count as sharing a border.)

2. Problem: Show that any map is k-colorable if and only if any finite submap is k-colorable.

Hint: Given a map M, introduce a sentential language $\mathscr{L}$ having the following atomic sentences: for each region x and color i, an atomic sentence C_{xi} intuitively saying that x has color i; for any two regions x and y, an atomic sentence B_{xy} saying that x and y share a border.

Devise a set Γ saying which regions share a border, that each region is to be assigned one and only one color, and that no two regions sharing a border can have the same color. Use the compactness theorem to obtain a k-coloring for the whole map.

3. Definition: A *society* is a triple $\mathscr{S} = (B, G, K)$ where B and G are disjoint sets (the "boys" and the "girls"), B is *finite* and K is a binary relation, "x knows y," with $\operatorname{dom} K \subseteq B$ and $\operatorname{rng} K \subseteq G$.

4. Definition: A society $\mathscr{S}$ satisfies the *happiness principle* if and only for every finite subset B_0 of B of cardinality k the set

$$\{g \in G : \exists b(b \in B_0 \wedge Kbg)\}$$

has cardinality $\geq k$.

A society satisfies the happiness principle if and only if whenever k boys give a party and each invites every girls he knows, at least k girls are invited. In particular, each boys knows at least one girl. Given a subset $B' \subseteq B$, let $K[B']$ be the point-wise image of B' under K, i.e., the set $\{g \in G : \exists b(b \in B' \wedge Kbg)\}$. Then the happiness condition can be formulated by saying that the cardinality of $K[B']$ must be greater than or equal to that of B'.

5. Definition: A *perfect matching* is a one-one correspondence f between B and G such that if $f(b) = g$ then Kbg.

A perfect matching is a way for each boy to dance with a girl he knows in such a way that no girl dances with more than one boy and no boy with more than one girl.

6. Problem: Every society satisfying the happiness principle has a perfect matching.

Hint: by induction on the cardinality n of B. The basis is obvious. For the inductive step, consider the following two cases:

(i) There is k with $0 < k < n$, and a subset B_0 of B of cardinality k such that the cardinality of

$$G_0 = \{g \in G : \exists b(b \in B_0 \wedge Kbg)\} = K[B_0]$$

is *exactly* k. Then $\mathscr{S}_0 = (B_0, G_0, K)$ is a society satisfying the happiness principle (why?). Apply the inductive hypothesis to obtain a perfect matching f_0 for $\mathscr{S}_0$. To obtain a perfect matching for all of $\mathscr{S}$, consider $B_1 = B \setminus B_0$ and $G_1 = G \setminus G_0$. Then $\mathscr{S}_1 = (B_1, G_1, K)$ is also a society satisfying the happiness principle (because if $B' \subseteq B_1$ is such that the cardinality of $K[B'] \setminus G_0$ is less than the cardinality of B' then $B_0 \cup B'$ is a counterexample to the happiness principle in $\mathscr{S}$). Therefore, $\mathscr{S}_1$ has a perfect matching f_1 by the inductive hypothesis, and $f_0 \cup f_1$ is a perfect matching for $\mathscr{S}$.

(*ii*) No such k and B_0 exist. Then for every B_0 the set $K[B_0]$ has cardinality at least $k+1$. Pick a $b_0 \in B$ and $g_0 \in G$ such that Kb_0g_0. If we put $B_0 = B \setminus \{b_0\}$ and $G_0 = G \setminus \{g_0\}$ then $\mathscr{S}_0 = (B_0, G_), K$ is a society satisfying the happiness principle, and so it has a perfect matching f_0 by inductive hypothesis. A perfect matching f can be obtained from f_0 by putting $f(b_0) = g_0$.

7. Definition: A *generalized society* is a triple $\mathscr{S} = (B, G, K)$ where B is allowed to be a countably infinite set. The society $\mathscr{S}$ satisfies the *normality principle* if every $b \in B$ is related by K to at most finitely many $g \in G$.

8. Problem: Show that every generalized society satisfying the happiness and normality principles has a perfect matching. *Hint*: use propositional compactness.

9. Problem: The normality assumption in the previous problem is necessary: show that there is a generalized society satisfying the happiness principle that has no perfect matching.

10. Definition: A relation R over a set X is *well-founded* if and only if there are no infinite descending chains in R, i.e., if there are no $x_0, x_1, x_2, \ldots$ in X such that $\ldots x_2 R x_1 R x_0$.

11. Problem: Assuming Zermelo-Fränkel set theory, ZF, is consistent, show that there are non-well-founded models of ZF, i.e., models $\mathfrak{A}$ such that $\ldots x_2 \in x_1 \in x_0$.

Problem Set IV: the game semantics

1. Definition: Given a structure $\mathfrak{A}$, an assignment s, and a formula φ (possibly containing free variables), the game $G(\mathfrak{A}, \varphi, s)$ is played between two players, Abelard and Eloïse, as follows:

- Each player can play the role of the *Verifier* or the *Falsifier*; the game starts with Abelard playing the Falsifier and Eloïse the Verifier.
- If the game is $G(\mathfrak{A}, Pt_1 \ldots, t_n, s)$ then whoever has the role of the Verifier wins if and only if $\mathfrak{A} \models Pt_1 \ldots, t_n[s]$; the Falsifier wins otherwise.
- if the game is $G(\mathfrak{A}, \sim\varphi, s)$ then Abelard and Eloïse switch roles and they play the game $G(\mathfrak{A}, \varphi, s)$.
- if the game is $G(\mathfrak{A}, \varphi \supset \psi, s)$ then the Verifier selects which one of the two games $G(\mathfrak{A}, \sim\varphi, s)$ and $G(\mathfrak{A}, \psi, s)$ is to be played next.
- If the game is $G(\mathfrak{A}, \forall x \varphi, s)$ then the Falsifier selects an $a \in A$ and the game proceeds as $G(\mathfrak{A}, \varphi, s(a/x))$.
- A *strategy* for a player is a prescription that selects a response for each of the other player's possible moves. A *winning strategy* for a player guarantees a win for that player if he or she follows the strategy.

2. Problem: Show that for every φ, the game $G(\mathfrak{A}, \varphi, s)$ is *determined*, in the sense that either the verifier has a winning strategy for the game, or the falsifier does (clearly they can't both have a winning strategy).

3. Problem: Show that $\mathfrak{A} \models \varphi[s]$ if and only if the Verifier has a winning strategy for $G(\mathfrak{A}, \varphi, s)$.

Hint: by induction on φ, obviously. The case for negation needs the previous problem, and for the Verifier to have a winning strategy for $G(\mathfrak{A}, \forall x \varphi, s)$, she must previously select a winning strategy for each game $G(\mathfrak{A}, \varphi, s(a/x))$, to use as a response to the Falsifier's move.

Problem Set V: definability

1. Definition: Let $\mathfrak{A}$ be a structure for a language $\mathscr{L}$. A subset B of $|\mathfrak{A}|$ is *definable* in $\mathscr{L}$ if and only if there is a formula $\varphi(x)$ with one free variable x such that for all assignments s, $\mathfrak{A} \models \varphi[s]$ iff $s(x) \in B$. In other words, for each assignment s:

$$B = \{a : \mathfrak{A} \models \varphi[s(a/x)]\}$$

2. Problem: Let $\mathscr{L}$ be the language containing a 2-place predicate symbol $<$ only (no other constants, function or predicate symbol — except of course $\doteq$). Let $\mathfrak{N}$ be the structure such that $|\mathfrak{N}| = \mathbb{N}$, and $<^{\mathfrak{N}} = \{(n, m) : n < m\}$. Prove the following:

1. $\{0\}$ is definable in $\mathfrak{N}$;
2. $\{1\}$ is definable in $\mathfrak{N}$;
3. $\{2\}$ is definable in $\mathfrak{N}$;
4. for each $n \in \mathbb{N}$, the set $\{n\}$ is definable in $\mathfrak{N}$;
5. every finite subset of $|\mathfrak{N}|$ is definable in $\mathfrak{N}$;
6. every co-finite subset of $|\mathfrak{N}|$ is definable in $\mathfrak{N}$ (where $B \subseteq \mathbb{N}$ is co-finite iff $\mathbb{N} \setminus B$ is finite).

3. Definition: An *automorphism* of a structure $\mathfrak{A}$ is an isomorphism of $\mathfrak{A}$ onto itself.

4. Problem: Show that for any structure $\mathfrak{A}$, if B is a definable subset of $|\mathfrak{A}|$, and h is an automorphism of $\mathfrak{A}$, then $B = \{h(a) : a \in B\}$ (i.e., B is fixed under h).

5. Problem: As in a previous problem, let $\mathscr{L}$ be the first-order language containing $<$ as its only predicate symbol (besides identity), and let $\mathfrak{N} = (\mathbb{N}, <)$. We know that all the finite or cofinite subsets of $\mathfrak{N}$ are definable: show that these are the *only* definable subsets of $\mathfrak{N}$.

Hint: First, let $\mathsf{prc}(x, y)$ be the $\mathscr{L}$-formula abbreviating "x is the immediate predecessor of y:"

$$x < y \wedge \neg\exists z(x < z \wedge z < y).$$

Now, to any definable subset of $\mathfrak{N}$ there corresponds a formula $\varphi(x)$ in $\mathscr{L}$. For any such φ, consider the sentence θ:

$$\exists x \forall y \forall z[(x < y \wedge x < z \wedge \mathsf{prc}(y, z) \wedge \varphi(y)) \supset \varphi(z)].$$

Show that $\mathfrak{N} \models \theta$ if and only if the subset of $\mathbb{N}$ defined by φ is either finite or cofinite.

Now, let $\mathfrak{M}$ be a non-standard model elementarily equivalent to $\mathfrak{N}$. If $a \in |\mathfrak{M}|$ is non-standard, let $b, c \in |\mathfrak{M}|$ be greater than a, and let b be the immediate predecessor of c. Then there is an automorphism

h of $\mathfrak{M}$ such that $h(b) = c$ (why?). Therefore, if b satisfies φ, so does c (why?). It follows that θ is true in $\mathfrak{M}$, and hence also in $\mathfrak{N}$. But this implies that the subset of $\mathfrak{N}$ defined by φ is either finite or co-finite.