Summary of Rules for All Systems

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Contents

1 Non-Modal Sentential Logic

1.1 Semantics for *SI*

1.1.1 Definitions

Semantical Entailment $\{\gamma_1 \ldots \gamma_n\} \vDash_{SI} \alpha$ if and only if for all interpretations **I**, if $\mathbf{v_I}(\gamma_1)$=**T**, and . . . and $\mathbf{v_I}(\gamma_n)$=**T**, then $\mathbf{v_I}(\alpha)$ =**T**

Semantical Equivalence α is semantically equivalent to β if and only if for all interpretations **I**, $\mathbf{v_I}(\alpha) = \mathbf{v_I}(\beta)$..

Validity $\vDash_{SI} \alpha$ if and only if on all interpretations **I**, $\mathbf{v_I}(\alpha) = \mathbf{T}$.

Semantical Consistency Γ is semantically consistent if and only if there is an interpretation *I* such that for all γ_i in Γ, $\mathbf{v_I}(\gamma_1)$=**T**, and . . . and $\mathbf{v_I}(\gamma_n)$=**T**.

Semantical Inconsistency Γ is semantically inconsistent if and only if there is no interpretation **I** such that for all γ_i in Γ, $\mathbf{v_I}(\gamma_1)$=**T**, and . . . and $\mathbf{v_I}(\gamma_n)$=**T**.

1.1.2 Primitive Rules

SR-TVA If α is a sentence-letter, then either $\mathbf{v_I}(\alpha)$=**T** or $\mathbf{v_I}(\alpha)$=**F**; it is not the case that $\mathbf{v_I}(\alpha)$=**T** and $\mathbf{v_I}(\alpha)$=**F**.

SR-⊥ For all I,$\mathbf{v_I}(\bot)$=**F** and $\mathbf{v_I}(\bot)\neq$**T**.

SR-∼ $\mathbf{v_I}(\sim\alpha)$=**T** if and only if $\mathbf{v_I}(\alpha)$=**F**; $\mathbf{v_I}(\sim\alpha)$=**F** if and only if $\mathbf{v_I}(\alpha)$=**T**.

SR-∧ $\mathbf{v_I}(\alpha \wedge \beta)$=**T** if and only if $\mathbf{v_I}(\alpha)$=**T** and $\mathbf{v_I}(\beta)$=**T**; $\mathbf{v_I}(\alpha \wedge \beta)$=**F** if and only if $_\mathbf{I}(\alpha)$=**F** or $\mathbf{v_I}(\beta)$=**F**.

SR-∨ $\mathbf{v_I}(\alpha \vee \beta)$=**T** if and only if $\mathbf{v_I}(\alpha)$=**T**, or $\mathbf{v_I}(\beta)$=**T**; $\mathbf{v_I}(\alpha \vee \beta)$=**F** if and only if $\mathbf{v_I}(\alpha)$=**F** and $\mathbf{v_I}(\beta)$=**F**.

SR-⊃ $\mathbf{v_I}(\alpha \supset \beta)$=**T** if and only if $\mathbf{v_I}(\alpha)$=**F** or $\mathbf{v_I}(\beta)$=**T**; $\mathbf{v_I}(\alpha \supset \beta)$=**F** if and only if $\mathbf{v_I}(\alpha)$=**T** and $\mathbf{v_I}(\beta)$=**F**.

SR-≡ $\mathbf{v_I}(\alpha \equiv \beta)$=**T** if and only if either $\mathbf{v_I}(\alpha)$=**T** and $\mathbf{v_I}(\beta)$=**T**, or $\mathbf{v_I}(\alpha)$=**F** and $\mathbf{v_I}(\beta)$=**F**; $\mathbf{v_I}(\alpha \equiv \beta)$=**F** if and only if either $\mathbf{v_I}(\alpha)$=**T** and $\mathbf{v_I}(\beta)$=**F**, or $\mathbf{v_I}(\alpha)$=**F** and $\mathbf{v_I}(\beta)$=**T**.

1.1.3 Derived Rules

Bivalence $\mathbf{v_I}(\alpha) = \mathbf{T} \bigvee \mathbf{v_I}(\alpha) = \mathbf{F}$.

Truth-Functionality $\neg(\mathbf{v_I}(\alpha) = \mathbf{T} \bigwedge \mathbf{v_I}(\alpha) = \mathbf{F})$.

1.2 Derivational Rules

1.2.1 Definitions

Derivability A sentence α is derivable in *SD* from a set of sentences $\{\gamma_1, \cdots, \gamma_n\}$, $\{\gamma_1, \cdots, \gamma_n\} \vdash_{SD} \alpha$, if and only if each member of the set is an assumption not in the scope of any other assumption, and α is a step in the immediate scope of those assumption(s).

Derivational Equivalence Two sentences α and β are *derivationally equivalent* in *SD* if and only if $\{\alpha\} \vdash_{SD} \beta$ and $\{\beta\} \vdash_{SD} \alpha$.

Theoremhood A sentence α is a theorem of *SD* if and only if there is a derivation of α in *SD* with no undischarged assumptions.

Derivational Consistency A set of sentences Γ of *SL* is derivationally consistent (d-consistent) in *SD*, if and only if it is not the case that there is a sentence α of *SL* such that $\Gamma \vdash_{SD} \alpha$ and $\Gamma \vdash_{SD} \sim\alpha$.

Derivational Inconsistency A set of sentences which is not derivationally consistent in *SD* is derivationally inconsistent in *SD*.

1.2.2 Primitive Rules

Reiteration

α		Already Derived
$\vdots$		
α		R
	β	Assumption
	α	R

Negation Introduction

	α	Assumption
	$\vdots$	
	β	
	$\vdots$	
	$\sim\beta$	
$\sim\alpha$		$\sim$ I

Negation Elimination

	$\sim\alpha$	Assumption
	$\vdots$	
	β	
	$\vdots$	
	$\sim\beta$	
α		$\sim$ E

Conjunction Introduction

α	Already Derived
$\vdots$	
β	Already Derived
$\vdots$	
$\alpha \wedge \beta$	$\wedge$ I
$\vdots$	
$\beta \wedge \alpha$	$\wedge$ I

Conjunction Elimination

$\alpha \wedge \beta$	Already Derived
$\vdots$	
α	$\wedge$ E
$\vdots$	
β	$\wedge$ E

Disjunction Introduction

α	Already Derived
$\vdots$	
$\alpha \vee \beta$	$\vee$ I
$\vdots$	
$\beta \vee \alpha$	$\vee$ I

Disjunction Elimination

$\alpha \vee \beta$	Already Derived
$\quad \alpha$	Assumption
$\quad \vdots$	
$\quad \gamma$	
$\quad \beta$	Assumption
$\quad \vdots$	
$\quad \gamma$	
γ	$\vee$ E

Material Conditional Introduction

$\quad \alpha$	Assumption
$\quad \vdots$	
$\quad \beta$	
$\alpha \supset \beta$	$\supset$ I

Material Conditional Elimination

α	Already Derived
$\vdots$	
$\alpha \supset \beta$	
$\vdots$	
β	$\supset$ E

Material Biconditional Introduction

$\quad \alpha$	Assumption
$\quad \vdots$	
$\quad \beta$	
$\quad \beta$	Assumption
$\quad \vdots$	
$\quad \alpha$	
$\alpha \equiv \beta$	$\equiv$ I

Material Biconditional Elimination

α	Already Derived
$\vdots$	
$\alpha \equiv \beta$	
$\vdots$	
β	$\equiv$ E

β	Already Derived
$\vdots$	
$\alpha \equiv \beta$	
$\vdots$	
α	$\equiv$ E

***Falsum* Introduction**

α	Already Derived
$\vdots$	
$\sim\alpha$	Already Derived
$\bot$	$\bot$ I

***Falsum* Elimination**

$\bot$	Already Derived
α	$\bot$ E

1.2.3 Derived Rules

Negation Introduction-⊥

α	Assumption
$\vdots$	
$\bot$	
$\sim\alpha$	$\sim$ I-$\bot$

Negation Elimination-⊥

$$\begin{array}{l|ll} & \sim\alpha & \text{Assumption} \\ \hline & \vdots & \\ & \bot & \\ \alpha & & \sim \text{E-}\bot \end{array}$$

2 The *K* Systems

2.1 Semantics for *KI*

2.1.1 Definitions

Note All definitions are applicable to *KI* and stronger modal semantical systems.

Frame **Fr**= ⟨**W**,**R**⟩

Interpretation **I**=⟨**W**,**R**, **v**⟩

Semantical Entailment in a Frame Fr $\{\gamma_1, \gamma_2, \ldots, \gamma_n\} \vDash_{\mathbf{Fr}} \alpha$ just in case for any **I** based on **Fr** and any **w** in **W** in **Fr**, if $\mathbf{v_I}(\gamma_1, \mathbf{w})$=**T**, **vI**($\gamma_2$, **w**)=**T**, and $\cdots$, and $\mathbf{v_I}(\gamma_n, \mathbf{w})$=**T**, then $\mathbf{v_I}(\alpha, \mathbf{w})$=**T**.

Semantical Entailment Semantical entailment in all frames.

Semantical Equivalence in a Frame Fr α is semantically equivalent to β in frame **Fr** just in case $\{\alpha\} \vDash \beta$ and $\{\beta\} \vDash \alpha$.

Semantical Equivalence Semantical equivalence in all frames.

Validity in Frame Fr $\vDash_{\mathbf{Fr}} \alpha$ iff for all **w** in **W** in **Fr**, all **I** based on **Fr**, and all $\mathbf{v_I}$ in **I**, $\mathbf{v_I}(\alpha, \mathbf{w})$=**T**

Validity Validity in all frames.

Semantical Consistency Γ is semantically consistent in *KI* if and only if there is an interpretation **I** based on a *KI* frame **Fr** in which there is a world **w** in the set of worlds **W** such that for all γ_i in Γ, $\mathbf{v_I}(\gamma_1, \mathbf{w})$=**T**, and . . . and $\mathbf{v_I}(\gamma_n, \mathbf{w})$=**T**.

Semantical Inconsistency Γ is semantically inconsistent in *KI* if and only if there is no interpretation **I** based on a frame **Fr** in which there is a world **w** in the set of worlds **W** such that for all γ_i in Γ, $\mathbf{v_I}(\gamma_1, \mathbf{w})$=**T**, and . . . and $\mathbf{v_I}(\gamma_n, \mathbf{w})$=**T**.

2.1.2 Primitive Rules

SR-TVA If α is a sentence-letter, then either $\mathbf{v_I}(\alpha, \mathbf{w})$=**T** or $\mathbf{v_I}(\alpha, \mathbf{w})$=**F**, but not both.

SR-⊥ For all **I** and all **w** in **I**, $\mathbf{v_I}(\bot, \mathbf{w})$=**F**, and $\mathbf{v_I}(\bot, \mathbf{w})$≠**T**.

SR-∼ $\mathbf{v_I}(\sim\alpha, \mathbf{w})$=**T** if and only if $\mathbf{v_I}(\alpha, \mathbf{w})$=**F**; $\mathbf{v_I}(\sim\alpha, \mathbf{w})$=**F** if and only if $\mathbf{v_I}(\alpha, \mathbf{w})$=**T**.

SR-$\wedge$ $\mathbf{v_I}(\alpha \wedge \beta, \mathbf{w})$=**T** if and only if $\mathbf{v_I}(\alpha, \mathbf{w})$=**T** and $\mathbf{v_I}(\beta, \mathbf{w})$=**T**; $\mathbf{v_I}(\alpha \wedge \beta, \mathbf{w})$=**F** if and only if $\mathbf{v_I}(\alpha, \mathbf{w})$=**F** or $\mathbf{v_I}(\beta, \mathbf{w})$=**T**.

SR-$\vee$ $\mathbf{v_I}(\alpha \vee \beta, \mathbf{w})$=**T** if and only if $\mathbf{v_I}(\alpha, \mathbf{w})$=**T** or $\mathbf{v_I}(\beta, \mathbf{w})$=**T**; $\mathbf{v_I}(\alpha \vee \beta, \mathbf{w})$=**F** if and only if $\mathbf{v_I}(\alpha, \mathbf{w})$=**F** and $\mathbf{v_I}(\beta, \mathbf{w})$=**F**.

SR-$\supset$ $\mathbf{v_I}(\alpha \supset \beta, \mathbf{w})$=**T** if and only if either $\mathbf{v_I}(\alpha, \mathbf{w})$=**F** or $\mathbf{v_I}(\beta, \mathbf{w})$=**T**; $\mathbf{v_I}(\alpha \supset \beta, \mathbf{w})$=**F** if and only if $\mathbf{v_I}(\alpha, \mathbf{w})$=**T** and $\mathbf{v_I}(\beta, \mathbf{w})$=**F**.

SR-$\equiv$ $\mathbf{v_I}(\alpha \equiv \beta, \mathbf{w})$=**T** if and only if either $\mathbf{v_I}(\alpha, \mathbf{w})$=**T** and $\mathbf{v_I}(\beta, \mathbf{w})$=**T**, or $\mathbf{v_I}(\alpha, \mathbf{w})$=**F** and $\mathbf{v_I}(\beta, \mathbf{w})$=**F**; $\mathbf{v_I}(\alpha \equiv \beta, \mathbf{w})$=**F** if and only if either $\mathbf{v_I}(\alpha, \mathbf{w})$=**T** and $\mathbf{v_I}(\beta, \mathbf{w})$=**F**, or $\mathbf{v_I}(\alpha, \mathbf{w})$=**F** and $\mathbf{v_I}(\beta, \mathbf{w})$=**T**.

SR-$\Diamond$ $\mathbf{v_I}(\Diamond\alpha, \mathbf{w})$=**T** if and only if $\mathbf{v_I}(\alpha, \mathbf{w}_i)$=**T** at some world $\mathbf{w}_i$ in **I** such that $\mathbf{Rww}_i$; $\mathbf{v_I}(\Diamond\alpha, \mathbf{w})$=**F** if and only if $\mathbf{v_I}(\alpha, \mathbf{w}_i)$=**F** at all worlds $\mathbf{w}_i$ in **I** such that $\mathbf{Rww}_i$.

SR-$\Box$ $\mathbf{v_I}(\Box\alpha, \mathbf{w})$=**T** if and only if $\mathbf{v_I}(\alpha, \mathbf{w}_i)$=**T** at all worlds $\mathbf{w}_i$ in **I** such that $\mathbf{Rww}_i$; $\mathbf{v_I}(\Box\alpha, \mathbf{w})$=**F** if and only if $\mathbf{v_I}(\alpha, \mathbf{w}_i)$=**F** at some world $\mathbf{w}_i$ in **I** such that $\mathbf{Rww}_i$.

SR-$⥽$ $\mathbf{v_I}(\alpha ⥽ \beta, \mathbf{w})$=**T** if and only if either $\mathbf{v_I}(\alpha, \mathbf{w}_i)$=**F** or $\mathbf{v_I}(\beta, \mathbf{w}_i)$=**T** at all worlds $\mathbf{w}_i$ in **I** such that $\mathbf{Rww}_i$; $\mathbf{v_I}(\alpha ⥽ \beta, \mathbf{w})$=**F** if and only if both $\mathbf{v_I}(\alpha, \mathbf{w}_i)$=**T** and $\mathbf{v_I}(\beta, \mathbf{w}_i)$=**F** at some world $\mathbf{w}_i$ in **I** such that $\mathbf{Rww}_i$.

2.1.3 Derived Rules

Modal Bivalence $\mathbf{v_I}(\alpha, \mathbf{w}) = \mathbf{T} \bigvee \mathbf{v_I}(\alpha, \mathbf{w}) = \mathbf{F}$.

Modal Truth-Functionality $\neg(\mathbf{v_I}(\alpha, \mathbf{w}) = \mathbf{T} \bigwedge \mathbf{v_I}(\alpha, \mathbf{w}) = \mathbf{F})$.

Closure of the '$\Box$' under *KI*-Entailment If $\{\gamma_1, \ldots, \gamma_n\} \vDash_{KI} \alpha$, then $\{\Box\gamma_1, \ldots, \Box\gamma_n\} \vDash_{KI} \Box\alpha$.

Necessitation of *KI*-valid Sentences If $\vDash_{KI} \alpha$, then $\vDash_{KI} \Box\alpha$.

2.2 Derivational Rules for *KD*

2.2.1 Definitions

Note All definitions are applicable to *KD* and stronger modal derivational systems.

Derivability A sentence α is derivable in *KD* from a set of sentences $\{\gamma_1, \cdots, \gamma_n\}$, $\{\gamma_1, \cdots, \gamma_n\} \vdash_{KD} \alpha$, if and only if each member of the set is an assumption not in the scope of any other assumption, and α is a step in the immediate scope of those assumption(s), and not within a restricted scope line.

Derivational Equivalence Two sentences α and β are *derivationally equivalent* in *KD* if and only if $\{\alpha\} \vdash_{KD} \beta$ and $\{\beta\} \vdash_{SD} \alpha$.

Theoremhood A sentence α is a theorem of *KD* if and only if there is a derivation of α in *KD* with no undischarged assumptions.

Derivational Consistency A set of sentences Γ of *SL* is derivationally consistent (d-consistent) in *KD*, if and only if it is not the case that there is a sentence α of *MSL* such that $\Gamma \vdash_{SD} \alpha$ and $\Gamma \vdash_{SD} {\sim}\alpha$.

Derivational Inconsistency A set of sentences which is not derivationally consistent in *KD* is derivationally inconsistent in *SD*.

2.2.2 Primitive Rules

Reiteration for Modal Derivations

α	Already Derived
$\vdots$	
α	R
$\quad \beta$	Assumption
$\quad \alpha$	R

Provided that any scope line crossed is not restricted.

Strict Reiteration for '$\Box$'

$\Box\alpha$	Already Derived
$\Box \quad \vdots$	
$\quad \alpha$	SR-$\Box$

$\Box$ Introduction

$\Box \quad \vdots$	
$\quad \alpha$	
$\Box\alpha$	$\Box$ I

Provided that α is not in the scope of any assumption within the strict scope line.

Strict Reiteration for $\sim\Diamond$

$\sim\Diamond\alpha$	Already Derived
$\Box \quad \vdots$	
$\quad \sim\alpha$	**SR**-$\sim\Diamond$

$\sim\Diamond$ Introduction

$\Box \quad \vdots$	
$\quad \sim\alpha$	
$\sim\Diamond\alpha$	$\sim\Diamond$ I

Provided that $\sim\alpha$ is not in the scope of any assumption within the strict scope line.

Strict Reiteration for '$\Diamond$'

$\Diamond\alpha$	Already Derived
$\Box$ α	SR-$\Diamond$
$\vdots$	

$\Diamond$ Elimination

$\Diamond\alpha$	Already Derived
$\Box$ α	SR-$\Diamond$
$\vdots$	
β	
$\Diamond\beta$	$\Diamond$ E

Provided that β is not in the scope of any assumption within the strict scope line.

Strict Reiteration for '⥽'

α ⥽ β	Already Derived
$\Box$ $\vdots$	
$\alpha \supset \beta$	SR-⥽

⥽ Introduction

$\Box$ α	
$\vdots$	
β	
α ⥽ β	⥽ I

Provided that β is not in the scope of any assumption within the strict scope line.

2.2.3 Derived Rules

Duality

$\sim\Diamond\alpha$	Already Derived
$\vdots$	
$\Box\sim\alpha$	Duality

$\Box\sim\alpha$	Already Derived
$\vdots$	
$\sim\Diamond\alpha$	Duality

$\sim\Box\alpha$	Already Derived
$\vdots$	
$\Diamond\sim\alpha$	Duality

$\sim\Box\alpha$	Already Derived
$\vdots$	
$\Diamond\sim\alpha$	Duality

3 The *D* Systems

3.1 Semantics for *DI*

As for *KI*, except with the restriction that **R** is **serial**:
$(\Pi\mathbf{w})(\mathbf{w} \in \mathbf{W} \rightarrow (\Sigma\mathbf{w}_i)(\mathbf{w}_i \in \mathbf{W} \wedge \mathbf{Rww}_i))$.

3.2 Derivational Rules for *DD*

3.2.1 Primitive Rule

Weak $\Diamond$ Introduction

$\Box$ $\vdots$	
α	
$\Diamond\alpha$	W$\Diamond$ I

Provided that α is not in the scope of any assumption within the strict scope line.

3.2.2 Derived Rule

Weak $\Diamond$ Introduction, Derived

$$\begin{array}{|ll} \vdots & \\ \Box\alpha & \\ \Diamond\alpha & \text{W}\Diamond\ \text{ID} \end{array}$$

Provided that α is not in the scope of any assumption within the strict scope line.

Other rules as for *KD*.

4 The *T* Systems

4.1 Semantics for *TI*

As for *KI*, except with the restriction that **R** is **reflexive**:
$(\Pi\mathbf{w})(\mathbf{w} \in \mathbf{W} \rightarrow \mathbf{Rww})$

4.2 Derivational Rules for *TD*

$\Box$ Elimination

$$\begin{array}{|ll} \Box\alpha & \\ \vdots & \\ \alpha & \Box\ \text{E} \end{array}$$

Strong $\Diamond$ Introduction

$$\begin{array}{|ll} \alpha & \\ \vdots & \\ \Diamond\alpha & \text{S}\ \Diamond\ \text{I} \end{array}$$

$\prec$ Elimination

$$\begin{array}{|ll} \alpha & \\ \dots & \\ \alpha \prec \beta & \\ \vdots & \\ \beta & \prec\ \text{E} \end{array}$$

Other rules as for *DD*.

5 The *S4* Systems

5.1 Semantics for *S4I*

As for *KI*, except with the restriction that **R** is reflexive (see *TI*) and **transitive**:
$(\Pi\mathbf{w})(\Pi\mathbf{w}_i)\Pi\mathbf{w}_j)((((\mathbf{w} \in \mathbf{W} \wedge \mathbf{w}_i \in \mathbf{W}) \wedge \mathbf{w}_j \in \mathbf{W}) \wedge(\mathbf{Rww}_i \wedge \mathbf{Rw}_i\mathbf{w}_j)) \rightarrow \mathbf{Rww}_j)$.

5.2 Derivational Rules for *S4D*

5.2.1 Primitive Rules

Strict Reiteration for '□' (4)

| $\Box\alpha$ Already derived
| □| ⋮
| | □| ⋮
| | | α SR-□(4)

Strict Reiteration for '∼◊' (S4)

| $\sim\Diamond\alpha$ Already derived
| □| ⋮
| | □| ⋮
| | | $\sim\alpha$ SR-∼◊(4)

◊ Elimination (4)

| $\Diamond\alpha$
| □| α
| | ⋮
| | $\Diamond\beta$
| | ⋮
| | □| β
| | | ⋮
| | | γ
| $\Diamond\gamma$ ◊ E (4)

Provided that γ is not in the scope of any assumption within the restricted scope line.

Strict Reiteration for '⥽' (S4)

α ⥽ β	Already derived
□| $\vdots$	
□| □| $\vdots$	
$\alpha \supset \beta$	SR-⥽(4)

5.2.2 Derived Rule

Strict Reiteration for '□' (4), Fitch

$\Box\alpha$	Already derived
□| $\vdots$	
$\Box\alpha$	SR-□(4)F

Other rules as for *TD*.

6 The *B* Systems

6.1 Semantics for *BI*

As for *KI*, except with the restriction that **R** is *reflexive* (see *TI*) and **symmetrical**: $(\Pi\mathbf{w})(\Pi\mathbf{w}_i)((\mathbf{w} \in \mathbf{W} \wedge \mathbf{w}_i \in \mathbf{W}) \wedge \mathbf{Rww}_i) \rightarrow \mathbf{Rw}_i\mathbf{w})$.

6.2 Derivational Rules for *BD*

Strict Reiteration (B)

α	Already derived
□| $\vdots$	
$\Diamond\alpha$	SR (B)

Strict Reiteration for '∼□' (B)

$\sim\alpha$	Already derived
□| $\vdots$	
$\sim\Box\alpha$	SR-∼□ (B)

Other rules as for *TD*.

7 The *S5* Systems

7.1 Semantical Rules for *S5I*

As for *KI*, except with the restriction that **R** is *reflexive* (see *TI*) and **euclidean**:
$(\Pi\mathbf{w})(\Pi\mathbf{w}_i)(\Pi\mathbf{w}_j)((((\mathbf{w} \in \mathbf{W} \bigwedge \mathbf{w}_i \in \mathbf{W}) \wedge \mathbf{w}_j \in \mathbf{W}) \bigwedge(\mathbf{Rww}_i \wedge \mathbf{Rww}_j)) \rightarrow \mathbf{Rw}_i\mathbf{w}_j)$.

Alternatively, as for *KI*, except with the restriction that **R** is an **equivalence** relation: **R** is reflexive, transitive, and symmetrical.

Alternatively, as for **KI**, except with the restriction that **R** is a **universal** relation:
$(\Pi\mathbf{w})(\Pi\mathbf{w}_i)((\mathbf{w} \in \mathbf{W} \bigwedge \mathbf{w}_i \in \mathbf{W}) \rightarrow \mathbf{Rww}_i)$.

Alternatively, the semantical system of Carnap.

$\mathbf{v_I}(\Box\alpha)$=**T** if and only if $(\Pi\mathbf{I})\mathbf{v_I}(\alpha)$=**T**; $\mathbf{v_I}(\Box\alpha)$=**F** if and only if $(\Sigma\mathbf{I})\mathbf{v_I}(\alpha)$=**F**.

$\mathbf{v_I}(\Diamond\alpha)$=**T** if and only if $(\Sigma\mathbf{I})\mathbf{v_I}(\alpha)$=**T**; $(\mathbf{v_I}(\Diamond\alpha)$=**F** if and only if $(\Pi\mathbf{I})\mathbf{v_I}(\alpha)$=**F**.

7.2 Derivational Rules for S5D

7.2.1 Primitive Rules

Strict Reiteration for '◊' (5)

	$\Diamond\alpha$	Already derived
□	⋮	
	$\Diamond\alpha$	SR-□(5)

7.2.2 Derived Rule

Strict Reiteration for Modalities

	α	Already derived
□	⋮	
	α	SR(M)

Where α is a sentence whose main operator is modal

Other rules as for *S4D* and *BD*.

8 Non-Modal Predicate Logic

8.1 Semantical Rules for *PI*

8.1.1 Definitions

- **Fr**=⟨**D**⟩.

- $\mathbf{I}=\langle\mathbf{D},\mathbf{v}\rangle$.
- Semantical entailment and validity as for *SI*.

8.1.2 Semantical Rules

SR-TVA If α is a sentence-letter, then either $\mathbf{v_I}(\alpha)=\mathbf{T}$ or $\mathbf{v_I}(\alpha)=\mathbf{F}$; it is not the case that $\mathbf{v_I}(\alpha)=\mathbf{T}$ and $\mathbf{v_I}(\alpha)=\mathbf{F}$.

SR-⊥ For all I,$\mathbf{v_I}(\bot)=\mathbf{F}$ and $\mathbf{v_I}(\bot)\neq\mathbf{T}$.

SR-Pred $\mathbf{v_I}(\mathbf{P}^n\mathbf{t}_1\ldots\mathbf{t}_n)=\mathbf{T}$ if and only if $\langle\mathbf{v_I}(\mathbf{t}_1),\ldots,\mathbf{v_I}(\mathbf{t}_n)\rangle\in\mathbf{v_I}(\mathbf{P}^n)$; $\mathbf{v_I}(\mathbf{P}^n\mathbf{t}_1\ldots\mathbf{t}_n)=\mathbf{F}$ if and only if $\neg(\langle\mathbf{v_I}(\mathbf{t}_1),\ldots,\mathbf{v_I}(\mathrm{t}_n)\rangle\in\mathbf{v_I}(\mathbf{P}^n))$

SR-∼ $\mathbf{v_I}(\sim\alpha)=\mathbf{T}$ if and only if $\mathbf{v_I}(\alpha)=\mathbf{F}$; $\mathbf{v_I}(\sim\alpha)=\mathbf{F}$ if and only if $\mathbf{v_I}(\alpha)=\mathbf{T}$.

SR-∧ $\mathbf{v_I}(\alpha\wedge\beta)=\mathbf{T}$ if and only if $\mathbf{v_I}(\alpha)=\mathbf{T}$ and $\mathbf{v_I}(\beta)=\mathbf{T}$; $\mathbf{v_I}(\alpha\wedge\beta)=\mathbf{F}$ if and only if $_\mathbf{I}(\alpha)=\mathbf{F}$ or $\mathbf{v_I}(\beta)=\mathbf{F}$.

SR-∨ $\mathbf{v_I}(\alpha\vee\beta)=\mathbf{T}$ if and only if $\mathbf{v_I}(\alpha)=\mathbf{T}$, or $\mathbf{v_I}(\beta)=\mathbf{T}$; $\mathbf{v_I}(\alpha\vee\beta)=\mathbf{F}$ if and only if $\mathbf{v_I}(\alpha)=\mathbf{F}$ and $\mathbf{v_I}(\beta)=\mathbf{F}$.

SR-⊃ $\mathbf{v_I}(\alpha\supset\beta)=\mathbf{T}$ if and only if $\mathbf{v_I}(\alpha)=\mathbf{F}$ or $\mathbf{v_I}(\beta)=\mathbf{T}$; $\mathbf{v_I}(\alpha\supset\beta)=\mathbf{F}$ if and only if $\mathbf{v_I}(\alpha)=\mathbf{T}$ and $\mathbf{v_I}(\beta)=\mathbf{F}$.

SR-≡ $\mathbf{v_I}(\alpha\equiv\beta)=\mathbf{T}$ if and only if either $\mathbf{v_I}(\alpha)=\mathbf{T}$ and $\mathbf{v_I}(\beta)=\mathbf{T}$, or $\mathbf{v_I}(\alpha)=\mathbf{F}$ and $\mathbf{v_I}(\beta)=\mathbf{F}$; $\mathbf{v_I}(\alpha\equiv\beta)=\mathbf{F}$ if and only if either $\mathbf{v_I}(\alpha)=\mathbf{T}$ and $\mathbf{v_I}(\beta)=\mathbf{F}$, or $\mathbf{v_I}(\alpha)=\mathbf{F}$ and $\mathbf{v_I}(\beta)=\mathbf{T}$.

SR-∀ $\mathbf{v_I}((\forall\mathbf{x})(\alpha(\mathbf{x}/\mathbf{u}))=\mathbf{T}$ (where $\mathbf{u}$ is free for $\mathbf{x}$ in $\alpha(\mathbf{x})$) if and only if for all $\mathbf{d}\in\mathbf{D}$, $\mathbf{v}_\mathrm{I}[\mathbf{d}/\mathbf{u}](\alpha(\mathbf{u}))=\mathbf{T}$; $\mathbf{v_I}((\forall\mathbf{x})(\alpha(\mathbf{x}/\mathbf{u}))=\mathbf{F}$ if and only if for some $\mathbf{d}\in\mathbf{D}$, $\mathbf{v_I}[\mathbf{d}/\mathbf{u}](\alpha(\mathbf{u}))=\mathbf{F}$.

SR-∃ $\mathbf{v_I}((\exists\mathbf{x})(\alpha(\mathbf{x}/\mathbf{u}))=\mathbf{T}$ (where $\mathbf{u}$ is free for $\mathbf{x}$ in $\alpha(\mathbf{x})$) if and only if for some $\mathbf{d}\in\mathbf{D}$, $\mathbf{v}_\mathrm{I}[\mathbf{d}/\mathbf{u}](\alpha(\mathbf{u}))=\mathbf{T}$; $\mathbf{v_I}((\exists\mathbf{x})(\alpha(\mathbf{x}/\mathbf{u}))=\mathbf{F}$ if and only if for all $\mathbf{d}\in\mathbf{D}$, $\mathbf{v_I}[\mathbf{d}/\mathbf{u}](\alpha(\mathbf{u}))=\mathbf{F}$.

$\mathbf{v_I}(\mathbf{t}_i=\mathbf{t}_j)=\mathbf{T}$ if and only if $\mathbf{v_I}(\mathbf{t}_i)=\mathbf{v_I}(\mathbf{t}_j)$;
$\mathbf{v_I}(\mathbf{t}_i=\mathbf{t}_j)=\mathbf{F}$ if and only if $\mathbf{v_I}(\mathbf{t}_i)\neq\mathbf{v_I}(\mathbf{t}_j)$;

8.2 Derivational Rules for *PD*

8.2.1 Primitive Rules

Universal Elimination

$(\forall\mathbf{x})\alpha(\mathbf{x}/\mathbf{u})$	Already Derived
⋮	
$\alpha(\mathbf{t}/\mathbf{u})$	∀ I

Universal Introduction

$$\begin{array}{l|l} & \vdots \\ & \mathbf{u} \,|\, \\ & \quad \vdots \\ & \quad \alpha(\mathbf{u}) \\ & (\forall \mathbf{x})\alpha(\mathbf{x}/\mathbf{u}) \qquad \forall\text{ I} \end{array}$$

Provided that: No sentence containing **u** is reiterated across the restricted scope line.

Existential Introduction

$$\begin{array}{|ll} \alpha(\mathbf{t}/\mathbf{u}) & \text{Already Derived} \\ \vdots & \\ (\exists \mathbf{x})\alpha(\mathbf{x}/\mathbf{u}) & \exists\text{ I} \end{array}$$

Existential Elimination

$$\begin{array}{|ll} (\exists \mathbf{x})\alpha(\mathbf{x}/\mathbf{u}) & \text{Already derived} \\ \mathbf{u} \,|\, \alpha(\mathbf{u}) & \text{Assumption} \\ \quad \vdots & \\ \quad \beta & \\ \beta & \exists\text{ E} \end{array}$$

Provided that: no sentence containing **u** is reiterated across the restricted scope line,
u does not occur in β.

Identity Introduction

$$\begin{array}{|ll} \vdots & \\ \mathbf{t} = \mathbf{t} & =\text{I} \end{array}$$

Identity Elimination

$$\begin{array}{|ll} \alpha(\mathbf{t}_i) & \text{Already Derived} \\ \vdots & \\ \mathbf{t}_i = \mathbf{t}_j & \text{Already Derived} \\ \vdots & \\ \alpha(\mathbf{t}_j/\mathbf{t}_i) & =\text{E} \end{array}$$

8.2.2 Derived Rules

Quantifier Exchange

$\sim(\forall \mathbf{x})\alpha(\mathbf{x}/\mathbf{u})$	Already Derived
$(\exists \mathbf{x})\sim\alpha(\mathbf{x}/\mathbf{u})$	$\sim\forall$

$\sim(\exists \mathbf{x})\alpha(\mathbf{x}/\mathbf{u})$	Already Derived
$(\forall \mathbf{x})\sim\alpha(\mathbf{x}/\mathbf{u})$	$\sim\exists$

$(\exists \mathbf{x})\sim\alpha(\mathbf{x}/\mathbf{u})$	Already Derived
$\sim(\forall \mathbf{x})\alpha(\mathbf{x}/\mathbf{u})$	$\exists\sim$

$(\forall \mathbf{x})\sim\alpha(\mathbf{x}/\mathbf{u})$	Already Derived
$\sim(\exists \mathbf{x})\alpha(\mathbf{x}/\mathbf{u})$	$\forall\sim$

Other rules as for *SD*.

9 Non-Modal Free Predicate Logic

9.1 Semantics for *FPI*

9.1.1 Definitions

- $\mathbf{I} = \{\mathbf{D}, \mathbf{D}', \mathbf{v}\}$
- $\mathbf{D} \neq \varnothing$
- $\mathbf{D} \cap \mathbf{D}' = \varnothing$

9.1.2 Special Semantical Rules

- $\mathbf{v_I}(\mathbf{a}) \in \mathbf{D} \cup \mathbf{D}'$
- $\mathbf{v_I}(\mathbf{u}) \in \mathbf{D} \cup \mathbf{D}'$
- $\mathbf{v}(\mathbf{P}^n) \subseteq (\mathbf{D} \cup \mathbf{D}')^n$
- $\mathbf{v_I}(\mathrm{E}) = \mathbf{D}^1$
- $\mathbf{v_I}((\forall \mathbf{x})(\alpha(\mathbf{x}/\mathbf{u})) = \mathbf{T}$ (where $\mathbf{u}$ is free for $\mathbf{x}$ in $\alpha(\mathbf{x})$) if and only if for all $\mathbf{d} \in \mathbf{D}$, $\mathbf{v}_{\mathrm{I}}[\mathbf{d}/\mathbf{u}](\alpha(\mathbf{u})) = \mathbf{T}$; $\mathbf{v_I}((\forall \mathbf{x})(\alpha(\mathbf{x}/\mathbf{u})) = \mathbf{F}$ if and only if for some $\mathbf{d} \in \mathbf{D}$, $\mathbf{v}_{\mathrm{I}}[\mathbf{d}/\mathbf{u}](\alpha(\mathbf{u})) = \mathbf{F}$.
- $\mathbf{v_I}((\exists \mathbf{x})(\alpha(\mathbf{x}/\mathbf{u})) = \mathbf{T}$ (where $\mathbf{u}$ is free for $\mathbf{x}$ in $\alpha(\mathbf{x})$) if and only if for some $\mathbf{d} \in \mathbf{D}$, $\mathbf{v}_{\mathrm{I}}[\mathbf{d}/\mathbf{u}](\alpha(\mathbf{u})) = \mathbf{T}$; $\mathbf{v_I}((\exists \mathbf{x})(\alpha(\mathbf{x}/\mathbf{u})) = \mathbf{F}$ if and only if for all $\mathbf{d} \in \mathbf{D}$, $\mathbf{v}_{\mathrm{I}}[\mathbf{d}/\mathbf{u}](\alpha(\mathbf{u})) = \mathbf{F}$.
- $\mathbf{v_I}(\mathbf{t}_i = \mathbf{t}_j) = \mathbf{T}$ if and only if $\mathbf{v_I}(\mathbf{t}_i) = \mathbf{v_I}(\mathbf{t}_j)$;
  $\mathbf{v_I}(\mathbf{t}_i = \mathbf{t}_j) = \mathbf{F}$ if and only if $\mathbf{v_I}(\mathbf{t}_i) \neq \mathbf{v_I}(\mathbf{t}_j)$;

9.2 Derivational Rules for *FPD*

Universal Elimination for Parameters (*FPL*)

$$\begin{array}{l|l|ll} & \mathbf{v} & (\forall \mathbf{x})\alpha(\mathbf{x}/\mathbf{u}) & \text{Already Derived} \\ & & \vdots & \\ & & \alpha(\mathbf{v}/\mathbf{u}) & \forall\text{ E (Parameters)} \end{array}$$

Universal Elimination for Constants (*FPL*)

$$\begin{array}{l|ll} & (\forall \mathbf{x})\alpha(\mathbf{x}/\mathbf{u}) & \text{Already derived} \\ & \vdots & \\ & \text{E}\mathbf{a} & \text{Already derived} \\ & \vdots & \\ & \alpha(\mathbf{a}/\mathbf{u}) & \forall\text{ E (Constants)} \end{array}$$

Existential Introduction for Parameters

$$\begin{array}{l|l|ll} & \mathbf{v} & \alpha(\mathbf{v}/\mathbf{u}) & \text{Already Derived} \\ & & \vdots & \\ & & (\exists \mathbf{x})\alpha(\mathbf{x}/\mathbf{u}) & \exists\text{ I (Parameters)} \end{array}$$

Barrier Removal (*FPL*)

$$\begin{array}{l|l|ll} & \mathbf{u} & \vdots & \\ & & \alpha & \text{BR} \\ & \alpha & & \end{array}$$

Provided that: **u** does not occur in α,
α does not lie in the scope of any undischarged assumption.

Existential Introduction for Constants

$$\begin{array}{l|ll} & \alpha(\mathbf{a}/\mathbf{u}) & \text{Already Derived} \\ & \vdots & \\ & \text{E}\mathbf{a} & \\ & \vdots & \\ & (\exists \mathbf{x})\alpha(\mathbf{x}/\mathbf{u}) & \exists\text{ I (Constants)} \end{array}$$

Other rules as for *PD*.

10 Systems *Q1R-x*

10.1 Semantical Rules for *Q1RI-x*

10.1.1 Definitions

- $\mathbf{I} = \{\mathbf{W}, \mathbf{R}, \mathbf{D}, \mathbf{q}, \mathbf{v}\}$
- $\mathbf{W} \neq \varnothing$
- $\mathbf{D} \neq \varnothing$
- $\mathbf{q}(\mathbf{w}) \subseteq \mathbf{D}$.
- $\mathbf{D}^{\mathbf{w}} = \mathbf{q}(\mathbf{w})$

10.1.2 Special Semantical Rules

- $\mathbf{v_I}(\mathbf{a},\mathbf{w}) \in \mathbf{D}$
- $\mathbf{v_I}(\mathbf{u},\mathbf{w}) \in \mathbf{D}$
- $(\Pi\mathbf{w}_i)(\Pi\mathbf{w}_j)((\mathbf{w}_i\in \mathbf{W} \wedge \mathbf{w}_j\in \mathbf{W}) \rightarrow \mathbf{v_I}(\mathbf{t}, \mathbf{w}_i)= \mathbf{v_I}(\mathbf{t}, \mathbf{w}_j))$
- $(\Pi\mathbf{w}_i)(\Pi\mathbf{w}_j)((\mathbf{w}_i\in \mathbf{W} \wedge \mathbf{w}_j\in \mathbf{W}) \rightarrow \mathbf{v_I}(\mathbf{t}, \mathbf{w}_i)= \mathbf{v_I}(\mathbf{t}, \mathbf{w}_j))$
- $\mathbf{v_I}(\mathbf{P}^n,\mathbf{w}) \subseteq \mathbf{D}^n$
- $\mathbf{v_I}(\mathbf{P}^n\mathbf{t}_1\ldots\mathbf{t}_n,\mathbf{w})=\mathbf{T}$ if and only if $\langle\mathbf{v_I}(\mathbf{t}_1,\mathbf{w}),\ldots,\mathbf{v_I}\mathbf{t}_1,\mathbf{w})\rangle \in \mathbf{v_I}(\mathbf{P}^n,\mathbf{w})$;
  $\mathbf{v_I}(\mathbf{P}^n\mathbf{t}_1\ldots\mathbf{t}_n, \mathbf{w})=\mathbf{F}$ if and only if $\neg(\langle\mathbf{v_I}(\mathbf{t}_1,\mathbf{w}),\ldots,\mathbf{v_I}(\mathrm{t}_n,\mathbf{w})\rangle \in \mathbf{v_I}(\mathbf{P}^n,\mathbf{w}))$
- $\mathbf{v_I}(\mathrm{E},\mathbf{w}) = \mathbf{D}^{\mathbf{w}1}$
- $\mathbf{v}_I(\mathbf{t}_i=\mathbf{t}_j,\mathbf{w})=\mathbf{T}$ if and only if $\mathbf{v}_I(\mathbf{t}_i,\mathbf{w})=\mathbf{v}_I(\mathbf{t}_j,\mathbf{w})$;
  $\mathbf{v}_I(\mathbf{t}_i=\mathbf{t}_j,\mathbf{w})=\mathbf{F}$ if and only if $\neg(\mathbf{v}_I(\mathbf{t}_i,\mathbf{w})=\mathbf{v}_I(\mathbf{t}_j,\mathbf{w}))$
- $\mathbf{v_I}((\forall\mathbf{x})(\alpha(\mathbf{x}/\mathbf{u}), \mathbf{w})=\mathbf{T}$ (where $\mathbf{u}$ is free for $\mathbf{x}$ in $\alpha(\mathbf{x})$) if and only if for all $\mathbf{d} \in \mathbf{D}^{\mathbf{w}}$, $\mathbf{v}_{\mathrm{I}}[\mathbf{d}/\mathbf{u}](\alpha(\mathbf{u}), \mathbf{w}) = \mathbf{T}$;
  $\mathbf{v_I}((\forall\mathbf{x})(\alpha(\mathbf{x}/\mathbf{u}), \mathbf{w})=\mathbf{F}$ if and only if for some $\mathbf{d} \in \mathbf{D}^{\mathbf{w}}$, $\mathbf{v}_{\mathrm{I}}[\mathbf{d}/\mathbf{u}](\alpha(\mathbf{u}), \mathbf{w}) = \mathbf{F}$.
- $\mathbf{v_I}((\exists\mathbf{x})(\alpha(\mathbf{x}/\mathbf{u}), \mathbf{w})=\mathbf{T}$ (where $\mathbf{u}$ is free for $\mathbf{x}$ in $\alpha(\mathbf{x})$) if and only if for some $\mathbf{d} \in \mathbf{D}^{\mathbf{w}}$, $\mathbf{v}_{\mathrm{I}}[\mathbf{d}/\mathbf{u}](\alpha(\mathbf{u}), \mathbf{w}) = \mathbf{T}$;
  $\mathbf{v_I}((\exists\mathbf{x})(\alpha(\mathbf{x}/\mathbf{u}))=\mathbf{F}$ if and only if for all $\mathbf{d} \in \mathbf{D}^{\mathbf{w}}$, $\mathbf{v}_{\mathrm{I}}[\mathbf{d}/\mathbf{u}](\alpha(\mathbf{u}), \mathbf{w}) = \mathbf{F}$.

10.2 Derivational Rules for *Q1RD-x*

Restricted Scope Line (*Q1R*)

$\square^n$ |

If the restricted scope line is not in the scope of any other restricted scope line, $n = 1$.
If the restricted scope line is in the immediate scope of a restricted scope line of index m, $n{=}m{+}1$.

Barrier Scope Line (*Q1R*)

$\square^n$ |
| ⋮
| u^n |
| | ⋮

If the barrier scope line is not in the scope of any restricted scope line, $n = 0$.
If the barrier scope line is in the immediate scope of a restricted scope line of index m, $n{=}m$.

Universal Elimination for Parameters (*Q1R*)

$\mathbf{v}^n$ $(\forall \mathbf{x})\alpha(\mathbf{x}/\mathbf{u}^n)$	Already Derived
⋮	
$\alpha(\mathbf{v}^n/\mathbf{u}^n)$	∀ I (*Q1R*)

Provided that: No sentence containing **v** is reiterated across the barrier scope line.

Universal Elimination for Constants (*Q1R*)

⋮	
$(\forall \mathbf{x})\alpha(\mathbf{x}/\mathbf{u})$	Already Derived
⋮	
E**a**	
⋮	
$\alpha(\mathbf{a}/\mathbf{u})$	∀ I (*Q1R*)

Universal Introduction (*Q1R*)

$$
\begin{array}{l|l}
 & \vdots \\
\mathbf{v}^n & \quad \vdots \\
 & \quad \alpha(\mathbf{v}^n/\mathbf{u}^n) \\
 & (\forall \mathbf{x})\alpha(\mathbf{x}/\mathbf{u}^n) \qquad \forall\ \mathrm{I}\ (\mathit{Q1R})
\end{array}
$$

Provided that: No sentence containing **u** is reiterated across the barrier scope line.

Existential Introduction for Parameters (*Q1R*)

$$
\begin{array}{l|ll}
\mathbf{v}^n & \alpha(\mathbf{v}^n/\mathbf{u}^n) & \text{Already Derived} \\
 & \vdots & \\
 & (\exists \mathbf{x})\alpha(\mathbf{x}/\mathbf{u}^n) & \exists\ \mathrm{I}\ (\mathit{Q1R})
\end{array}
$$

Provided that: No sentence containing **v** is reiterated across the barrier scope line.

Existential Introduction for Constants (*Q1R*)

$$
\begin{array}{|ll}
\alpha(\mathbf{a}/\mathbf{u}) & \text{Already Derived} \\
\vdots & \\
\mathrm{E}\mathbf{a} & \\
\vdots & \\
(\exists \mathbf{x})\alpha(\mathbf{x}/\mathbf{u}) & \exists\ \mathrm{I}\ (\mathit{Q1R})
\end{array}
$$

Provided that: No sentence containing **v** is reiterated across the barrier scope line.

Existential Elimination (*Q1R*)

$$
\begin{array}{l|ll}
 & (\exists \mathbf{x})\alpha(\mathbf{x}/\mathbf{u}^n) & \text{Already derived} \\
\mathbf{v}^n & \quad \alpha(\mathbf{v}^n/\mathbf{u}^n) & \text{Assumption} \\
 & \quad \vdots & \\
 & \quad \beta & \\
 & \beta & \exists\ \mathrm{E}
\end{array}
$$

Provided that: no sentence containing **v** is reiterated across the restricted scope line,
v does not occur in β.

Strict Reiteration for '='

$\mathbf{t}_i = \mathbf{t}_j$ Already Derived

$\Box^n$ $\vdots$

$\mathbf{t}_i = \mathbf{t}_j$ SR-=

Strict Reiteration for '∼ ='

$\sim\mathbf{t}_i = \mathbf{t}_j$ Already Derived

$\Box^n$ $\vdots$

$\sim\mathbf{t}_i = \mathbf{t}_j$ SR-∼ =

Other rules as for modal system *x*.

11 Systems *Q1RC-x*

11.1 Semantical Rules for *Q1RCI-x*

$\mathbf{Rw}_i\mathbf{w}_j \rightarrow \mathbf{D}^{\mathbf{w}_i} \subseteq \mathbf{D}^{\mathbf{w}_j}$ (Included In)

Other rules as for *Q1RI-x*

11.2 Derivational Rules for *Q1RCD-x*

Index Decrement for '∀' (*Q1RC)*

$\Box^n$ $\alpha(\underline{\mathbf{u}^n})$ Given

$\vdots$

$\alpha(\underline{\mathbf{u}^{n-1}})$ ID ∀ (*Q1RC)*

Barrier Removal for '∀' (*Q1RC*)

$\mathbf{u}^n$ $\vdots$

$\alpha(\underline{\mathbf{u}^{n-1}})$

$\alpha(\underline{\mathbf{u}^{n-1}})$ BR ∀ (*Q1RC)*

Provided that: Except for $\underline{\mathbf{u}^{n-1}}$, **u** does not occur in α,
α does not lie in the scope of any undischarged assumption.

Index Increment for '∃' (*Q1RC)*

$\Box^n$ $\vdots$

$\alpha(\overline{\mathbf{u}^{n-1}}/\mathbf{x})$

$\vdots$

$\alpha(\overline{\mathbf{u}^n}/\mathbf{x})$ II ∃

Barrier Crossing for ∃ (*Q1RC*)

$$\alpha(\overline{\mathbf{u}^{n-1}})$$
$$\begin{array}{l|l|ll} & \vdots & & \\ & \mathbf{u}^n & & \\ & & \vdots & \\ & & \alpha(\overline{\mathbf{u}^{n-1}}) & \text{BC } \exists \ (Q1RC) \end{array}$$

Other rules as for *Q1RD-x*

12 Systems *Q1RB-x*

12.1 Semantical Rules for *Q1RBI-x*

$\mathbf{Rw}_i\mathbf{w}_j \rightarrow \mathbf{D}^{\mathbf{w}_i} \supseteq \mathbf{D}^{\mathbf{w}_j}$ (Includes)

Other rules as for *Q1RI-x*

12.2 Derivational Rules for *Q1RBD-x*

Index Increment for '∀' (*Q1RB*)

$$\begin{array}{l|ll} \square^n & \alpha(\underline{\mathbf{u}^n}) & \text{Given} \\ & \vdots & \\ & \alpha(\underline{\mathbf{u}^{n+1}}) & \text{II } \forall \ (Q1RB) \end{array}$$

Barrier Crossing for ∀ (*Q1RB*)

$$\alpha(\underline{\mathbf{u}^{n-1}})$$
$$\begin{array}{l|l|ll} & \vdots & & \\ & \mathbf{u}^n & & \\ & & \vdots & \\ & & \alpha(\underline{\mathbf{u}^{n-1}}) & \text{BC } \forall \ (Q1RB) \end{array}$$

Index Decrement for '∃' (*Q1RB*)

$$\begin{array}{l|ll} \square^n & \alpha(\overline{\mathbf{u}^n}) & \text{Given} \\ & \vdots & \\ & \alpha(\overline{\mathbf{u}^{n-1}}) & \text{ID } \exists \ (Q1RB) \end{array}$$

Barrier Removal for '∃' (*Q1RB*)

$$\begin{array}{l|l} \mathbf{u}^n & \vdots \\ & \alpha(\overline{\mathbf{u}^{n-1}}) \qquad \text{BR } \exists \ (\textit{Q1RB}) \\ \multicolumn{2}{l}{\alpha(\overline{\mathbf{u}^{n-1}})} \end{array}$$

Provided that: Except for $\overline{\mathbf{u}^{n-1}}$, **u** does not occur in α,
α does not lie in the scope of any undischarged assumption.

Other rules as for *Q1RD-x*.

13 Systems *FQ1-x*

13.1 Semantical Rules for *FQ1I-x*

13.1.1 Definitions

- $\mathbf{I} = \{\mathbf{D}, \mathbf{D}', \mathbf{v}\}$
- $\mathbf{W} \neq \emptyset$
- $\mathbf{D} \neq \emptyset$
-

13.1.2 Special Semantical Rules

- $\mathbf{v_I}(\mathbf{a}) \in \mathbf{D} \cup \mathbf{D}'$
- $\mathbf{v_I}(\mathbf{u}) \in \mathbf{D} \cup \mathbf{D}'$
- $(\Pi \mathbf{w}_i)(\Pi \mathbf{w}_j)((\mathbf{w}_i \in \mathbf{W} \wedge \mathbf{w}_j \in \mathbf{W}) \rightarrow (\mathbf{v_I}(\mathbf{t}, \mathbf{w}_i) = (\mathbf{v_I}(\mathbf{t}, \mathbf{w}_j))$
- $\mathbf{v}(\mathbf{P}^n) \subseteq (\mathbf{D} \cup \mathbf{D}')^n$
- $\mathbf{v_I}(\mathrm{E}) = \mathbf{D}^1$
- $\mathbf{v_I}((\forall \mathbf{x})(\alpha(\mathbf{x}/\mathbf{u}), \mathbf{w}) = \mathbf{T}$ (where **u** is free for **x** in $\alpha(\mathbf{x})$) if and only if for all $\mathbf{d} \in \mathbf{D}$, $\mathbf{v}_\mathrm{I}[\mathbf{d}/\mathbf{u}](\alpha(\mathbf{u}), \mathbf{w}) = \mathbf{T}$; $\mathbf{v_I}((\forall \mathbf{x})(\alpha(\mathbf{x}/\mathbf{u}), \mathbf{w}) = \mathbf{F}$ if and only if for some $\mathbf{d} \in \mathbf{D}$, $\mathbf{v}_\mathrm{I}[\mathbf{d}/\mathbf{u}](\alpha(\mathbf{u}), \mathbf{w}) = \mathbf{F}$.
- $\mathbf{v_I}((\exists \mathbf{x})(\alpha(\mathbf{x}/\mathbf{u}), \mathbf{w}) = \mathbf{T}$ (where **u** is free for **x** in $\alpha(\mathbf{x})$) if and only if for some $\mathbf{d} \in \mathbf{D}$, $\mathbf{v}_\mathrm{I}[\mathbf{d}/\mathbf{u}](\alpha(\mathbf{u}), \mathbf{w}) = \mathbf{T}$; $\mathbf{v_I}((\exists \mathbf{x})(\alpha(\mathbf{x}/\mathbf{u}), \mathbf{w}) = \mathbf{F}$ if and only if for all $\mathbf{d} \in \mathbf{D}$, $\mathbf{v}_\mathrm{I}[\mathbf{d}/\mathbf{u}](\alpha(\mathbf{u}), \mathbf{w}) = \mathbf{F}$.
- $\mathbf{v_I}(\mathbf{t}_i = \mathbf{t}_j, \mathbf{w}) = \mathbf{T}$ if and only if $\mathbf{v_I}(\mathbf{t}_i, \mathbf{w}) = \mathbf{v_I}(\mathbf{t}_j, \mathbf{w})$;
  $\mathbf{v_I}(\mathbf{t}_i = \mathbf{t}_j, \mathbf{w}) = \mathbf{F}$ if and only if $\mathbf{v_I}(\mathbf{t}_i) \neq \mathbf{v_I}(\mathbf{t}_j, \mathbf{w})$;

13.2 Derivational Rules for *FQ1D-x*

As for both *Q1RCD-x* and *Q1RCB-x*.

14 Systems *Q1-x*

14.1 Semantical Rules for *Q1I-x*

14.1.1 Definitions

- $\mathbf{I} = \{\mathbf{W}, \mathbf{R}, \mathbf{D}, \mathbf{v}\}$
- $\mathbf{D} \neq \varnothing$
- $\mathbf{W} \neq \varnothing$

14.1.2 Special Semantical Rules

- $\mathbf{v_I}(\mathbf{a}) \in \mathbf{D}$
- $\mathbf{v_I}(\mathbf{u}) \in \mathbf{D}$
- $(\Pi \mathbf{w}_i)(\Pi \mathbf{w}_j)((\mathbf{w}_i \in \mathbf{W} \wedge \mathbf{w}_j \in \mathbf{W}) \rightarrow (\mathbf{v_I}(\mathbf{t}, \mathbf{w}_i) = (\mathbf{v_I}(\mathbf{t}, \mathbf{w}_j))$
- $\mathbf{v}(\mathbf{P}^n) \subseteq \mathbf{D}^n$
- $\mathbf{v_I}(\mathrm{E}) = \mathbf{D}^1$
- $\mathbf{v_I}((\forall \mathbf{x})(\alpha(\mathbf{x}/\mathbf{u}), \mathbf{w})\mathbf{=T}$ (where $\mathbf{u}$ is free for $\mathbf{x}$ in $\alpha(\mathbf{x})$) if and only if for all $\mathbf{d} \in \mathbf{D}$, $\mathbf{v}_\mathrm{I}[\mathbf{d}/\mathbf{u}](\alpha(\mathbf{u}), \mathbf{w}) = \mathbf{T}$; $\mathbf{v_I}((\forall \mathbf{x})(\alpha(\mathbf{x}/\mathbf{u}), \mathbf{w})\mathbf{=F}$ if and only if for some $\mathbf{d} \in \mathbf{D}$, $\mathbf{v}_\mathrm{I}[\mathbf{d}/\mathbf{u}](\alpha(\mathbf{u}), \mathbf{w}) = \mathbf{F}$.
- $\mathbf{v_I}((\exists \mathbf{x})(\alpha(\mathbf{x}/\mathbf{u}), \mathbf{w})\mathbf{=T}$ (where $\mathbf{u}$ is free for $\mathbf{x}$ in $\alpha(\mathbf{x})$) if and only if for some $\mathbf{d} \in \mathbf{D}$, $\mathbf{v}_\mathrm{I}[\mathbf{d}/\mathbf{u}](\alpha(\mathbf{u}), \mathbf{w}) = \mathbf{T}$; $\mathbf{v_I}((\exists \mathbf{x})(\alpha(\mathbf{x}/\mathbf{u}), \mathbf{w})\mathbf{=F}$ if and only if for all $\mathbf{d} \in \mathbf{D}$, $\mathbf{v}_\mathrm{I}[\mathbf{d}/\mathbf{u}](\alpha(\mathbf{u}), \mathbf{w}) = \mathbf{F}$.
- $\mathbf{v_I}(\mathbf{t}_i\mathbf{=t}_j, \mathbf{w})\mathbf{=T}$ if and only if $\mathbf{v_I}(\mathbf{t}_i, \mathbf{w})\mathbf{=}\mathbf{v_I}(\mathbf{t}_j, \mathbf{w})$;
  $\mathbf{v_I}(\mathbf{t}_i\mathbf{=t}_j, \mathbf{w})\mathbf{=F}$ if and only if $\mathbf{v_I}(\mathbf{t}_i) \neq \mathbf{v_I}(\mathbf{t}_j, \mathbf{w})$;

14.2 Derivational Rules for *Q1D-x*

Universal Elimination (*Q1D*)

$(\forall \mathbf{x})\alpha(\mathbf{x}/\mathbf{u})$	Already Derived
⋮	
$\alpha(\mathbf{a}/\mathbf{u})$	∀ I

Existential Introduction (*Q1D*)

$\alpha(\mathbf{a}/\mathbf{u})$	Already Derived
⋮	
$(\exists \mathbf{x})\alpha(\mathbf{x}/\mathbf{u})$	∃ I

Other rules as for *FQ1D-x*