Informal Introduction to Modal Logic

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1 Motivation for Modal Logic

One of the first things learned by beginning logic students is the definition of a **valid argument**. The standard definition of a valid argument runs along these lines: An argument from a set of premises to a conclusion is valid if and only if it is not possible for all the premises of the argument to be true and for the conclusion to be false. Alternatively, one might say that an argument is valid just in case it is that necessarily, if the premises are true, then the conclusion is true.

The concept of a valid argument lies at the heart of logic. The definition of the concept of validity, in turn, depends essentially on the **modal** concepts of possibility and necessity. These concepts are called "modal" because they indicate a "way" or "mode" in which the truth-values of the premises are connected with the truth-values of the conclusion.

One of the main tasks of symbolic logic is to represent the form of arguments, in such a way that their validity or invalidity can be determined using standardized techniques. One can use truth-tables, for example, to represent the validity or invalidity of arguments whose basic units are individual sentences.

In the logics that are generally taught in introductory logic courses, the properties of validity or invalidity are not represented in the symbolic language itself. There are symbols representing the "truth-functional" operators 'and', 'or', 'not', etc., but there is no symbol for 'therefore'. [1] In fact, the operators of truth-functional logic, as well as those of predicate logic, do not represent possibility or necessity at all. Standard logic is non-modal in this respect, even though it might be used to establish modal properties of arguments.

Modal logics are precisely those logical systems which contain modal operators. In the case of validity one might seek to build a logical language which contains an operator which is understood to express the modal property of validity. That is, it would contain a **modal operator**.

2 The Lewis Systems

Modern modal logic appeared in the early twentieth century, not long after modern non-modal logics had been popularized by Russell and Whitehead's

[1] Many logic students mistakenly render the "material conditional" into English as 'therefore', but this is a mistake.

Principia Mathematica.[2] A young philosophy instructor at UC Berkeley, C.I. Lewis, used *Principia* as a text. Lewis thought that Russell's description of the truth-functional conditional operator as "material implication" was misleading. He built several axiomatic systems featuring a modal operator he called "strict implication", which he thought better represents the relation between premise and conclusion in an argument.

One-place modal operators for possibility and necessity were part of the Lewis systems. The necessity operator can be understood as allowing the represention of the concept of necessary truth. "Strict implication", it turned out, expresses the same thing as necessarily true "material implication".

3 Semantics

Lewis's systems were laid down in axiomatic form. The earliest work on the systems was to prove theorems of the given systems which follow from their specific axioms. Very soon thereafter, the axiomatic systems were given interpretations. The most prominent kind of interpretation was with "matrices" that resemble truth-tables. A useful matrix for modal logic typically contained more than two values. Using matrices, logicians were able to get important results about the systems. They could determine which systems contain which other systems and whether a given axiom is independent of the other axioms.[3]

While the matrix system was useful, it provided no real insight into the meanings of modal sentences. It was Rudolf Carnap, writing in the mid 1940s, who first provided an intuitively plausible semantics for one of the Lewis systems, S5. In Carnap's semantical system, the truth-values of non-modal sentences are determined just as they are in truth-functional logic. A sentence whose main operator is a necessity operator is true if and only if the sentence it governs is a logical truth. Thus, if a sentence is true on all rows of its truth-table, then the sentence formed by prefixing a necessity operator to it is also true, and in fact is true on all rows of all truth-tables.

Carnap also provided a system for a predicate-logic version of S5. His semantics is of interest because of the way it interprets the syntax of predicate logic. Non-modal semantics interprets terms as standing for objects in a universe or domain of discourse. Predicates are interpreted as standing for sets of objects from the domain. Beside this "extensional" type of interpretation, Carnap developed an "intensional" interpretation suitable for the use of terms and predicates in modal contexts.

The intensional interpretation depends on the notion of a "state description". Carnap wrote that "the state description represents Leibniz's 'possible worlds' or Wittgenstein's possible states of affairs".[4] A term might stand for different objects in different state descriptions, so that its "intension" is a function

[2] The basic system in that book had been laid out by Frege in 1879, but had gone largely unnoticed.

[3] System S contains system S′ just in case all the theorems of S′ are theorems of S.

[4] *Meaning and Necessity*, 9-10.

from state descriptions to objects. A perdicate might have different extensions in different state descriptions, so that its "intension" is a function from state descriptions to sets of objects.

In the late 1950s, Carnap's semantics was generalized to the form in which it exists today.[5] The key notion in the semantics is that of a "possible world". In sentential logic, a possible world corresponds to a row of a truth-table. In predicate logic, a possible world corresponds (roughly) to an interpretation.

Carnap's system, in effect, took necessity to be truth at all possible worlds. This worked as a semantics for S5, but not for any of the weaker systems of Lewis and others. The innovation was to add to the semantics a two-place relation of "accessiblity" or "alternativeness" holding among the worlds themselves. Then a necessity sentence could be taken to be true just in case the sentence governed by the necessity operator is true at all accessible possible worlds. This generalization of the Carnapian semantics allowed Kripke and others to provide semantics for most of the known axiomatic systems of modal logic. It also made it easy to generate new systems. Most importantly, perhaps, it provided an intuitive way of understanding what the sentences of modal logic mean.

It should be noted that for a long time, modal logic was held in some disrepute, due to the criticisms of W.v.O. Quine. One of his objections was that valdity, implication, logical necessity, etc. are meta-logical notions which have no place in logic itself. Another was that the semantics for modal predicate logic requires the postulation of possible but non-existent objects. Quine believed that we should not commit ourselves to "possibilia", on the grounds that they do not have well-defined "identity conditions".

When generalized possible worlds semantics came on the scene, philosophers welcomed it as a powerful analytic tool and brushed Quine's objections aside. It is probably not coincidental that about this time there was a powerful shift away from the austere metaphysics of Quine and his Harvard colleague Nelson Goodman (not to mention the later Wittgenstein). There remains vigorous debate about the metaphysical status of possible worlds and objects in them. At one extreme, David Lewis advocated a modal realism, according to which each possible world is just as real as the one we call "actual".[6] At another, Michael Jubien has tried to treat modalities without appeal to possible worlds at all.[7]

4 Applications

From the time generalized possible worlds semantics was invented, and even before, philosophical logicians began to recognize that it has applications beyond the logical notions of implication and logical truth. Jaakko Hintikka recognized that Lewis's "necessity" operator could be interpreted either as a belief opera-

[5]The sematics is commonly attributed to Saul A. Kripke, but it was developed during the same period by Jaakko Hintikka and Stig Kanger.

[6]*On the Plurality of Worlds*, 1986.

[7]*Contemporary Metaphysics*, 1997, Chapter 8.

tor or a knowledge operator.[8] Arthur Prior had explored the use of the modal operators to represent modalities of time.[9] G. H. von Wright in 1951 had interpreted the modal operators in terms of obligation and permissibility.[10] All of these suggestions have been developed in great detail during the past few decades.

5 Natural Deduction Systems

In 1952, Frederick Fitch published a very influential logic text, *Symbolic Logic: An Introduction*. In this book, Fitch adapted the "natural deduction" systems of Gerhard Gentzen and others in a way that made it relatively easy to prove theorems and the validity of arguments syntatically. Fitch also provided rules for the Lewis systems S4 and S2.[11] Fitch's approach has been generalized to a number of other modal systems.

6 Plan of the Text

The aim of this text is twofold. The first aim is to acquaint the reader with the basic formal characteristics of a wide range of systems of modal logic. Each system will be introduced semantically. Fitch-style natural deduction rules will then be given and treated as shorthand for obtaining semantical results. In the first part of the text, a basic sentential modal system, K, will be laid out. In the second part, an number of other sentential systems will be considered. The third section is a brief introduction to predicate modal systems.

The second aim of the text is to explore the ways in which the various systems can be applied. We will consider modality in six different ways: (1) as a logic of necessary truth and falsehood, "alethic" modal logic; (2) as a logic of implication between a sentence and what follows from it, "implicational" modal logic; (3) as a logic of obligation and permission, "deontic" logic; (4) as a logic of knowledge, "epistemic" logic; (5) as a logic of belief, "doxastic" logic; (6) as a logic of time, "temporal logic".

[8] *Knowledge and Belief: An Introduction to the Logic of the Two Notions*, 1962.

[9] *Time and Modality*, 1957.

[10] "Deontic Logic", in *Mind*.

[11] These rules map very nicely onto the generalized possible worlds semantics, as will be seen. Lewis did not recognize the connection, however.