Module 8
T and Equivalent Systems

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January 2, 2014

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Some kinds of modalities demand that what is necessary at a world be true at that same world. This requirement is met by the *T*-systems. The axiomatic system is also known in the literature as *M*. It was first investigated by R. Feys and G. H. von Wright.[1]

The *T*-systems are extensions of the *D*-systems, and therefore are extensions of the *K*-systems as well. As with the other two families of systems, we will discuss the semantical system first, then the derivational system, then, briefly, the axiom system, and finally applications.

1 The Semantical System *TI*

The semantical system *TI* is just like the system *KI* except for the requirement that **R** be *reflexive*. That is, each world must be accessible to itself. We may express the reflexive character of a relation **R** as follows:

R is **reflexive** if and only if $(\Pi \mathbf{x})\mathbf{Rxx}$.

We can define a *TI*-frame as a set $\langle \mathbf{W}, \mathbf{R} \rangle$, such that :

$(\Pi \mathbf{w})(\mathbf{w} \in \mathbf{W} \rightarrow \mathbf{Rww})$.

A *TI* interpretation meets the following condition:

[1] Feys, 1927, "Les logiques nouvelles des modalités," *Revue Néoscholastique de Philosophie*, 40, 517-53, 41, 217-52. von Wright, *An Essay in Modal Logic*, 1951.

$(\Pi \mathbf{w})(\mathbf{BwI} \rightarrow \mathbf{Rww})$.

Implied by the reflexivity of the accessibility relation in *TI*-frames is one of the two characteristic consequence relations of the *T*-systems.

$\{\Box\alpha\} \vDash_{TI} \alpha$.

Proof. If $\mathbf{v_I}(\Box\alpha, \mathbf{w})$=$\mathbf{T}$, then for all worlds $\mathbf{w}_i$ accessible to $\mathbf{w}$, $\mathbf{v_I}(\alpha, \mathbf{w}_i)$=$\mathbf{T}$. By the reflexivity of $\mathbf{R}$, $\mathbf{v_I}(\alpha, \mathbf{w})$=$\mathbf{T}$. The proof of the entailment can also be given using a meta-logical derivation.

Semantical proof that: $\{\Box\alpha\} \vDash_{TI} \alpha$

1	$\mathbf{v_I}(\Box\alpha, \mathbf{w}) = \mathbf{T}$	Assumption
2	$(\Pi\mathbf{w})\mathbf{Rww}$	Reflexivity of $\mathbf{R}$
3	$\mathbf{Rww}$	2 Π E
4	$(\Pi\mathbf{w}_i)(\mathbf{Rww}_i \rightarrow \mathbf{v_I}(\alpha, \mathbf{w}_i) = \mathbf{T})$	1 **SR**-$\Box$
5	$\mathbf{Rww} \rightarrow \mathbf{v_I}(\alpha, \mathbf{w}) = \mathbf{T}$	4 Π E
6	$\mathbf{v_I}(\alpha, \mathbf{w}) = \mathbf{T}$	3 5 $\rightarrow$ E

The reflexivity of accessibility can be diagrammed using an arrow that loops back to the home world. Then we can use a modal truth-table to illustrate the characteristic *DI* consequence.

$$\begin{array}{c} \curvearrowleft \\ \mathbf{w} \\ \Box\alpha \\ \hline \mathbf{T} \\ \alpha \\ \hline \mathbf{T} \end{array}$$

The second characteristic consequence relation in *TI* is the dual of the first:

$\{\alpha\} \vDash_{TI} \Diamond\alpha$.

Here we will illustrate its proof using a modal truth-table.

$$\begin{array}{c} \curvearrowleft \\ \mathbf{w} \\ \alpha \\ \hline \mathbf{T} \\ \Diamond\alpha \\ \hline \mathbf{T} \end{array}$$

All *TI*-frames are *DI*-frames. If a world $\mathbf{w}$ belongs to an interpretation $\mathbf{I}$, then $\mathbf{w}$ is accessible to itself. If a world in an interpretation is accessible to itself, then there is a world (itself) which is accessible to it. More generally, any relation that is reflexive is also serial. The converse does not hold, since the accessible world required by seriality need not be the home world, so not all *DI*-frames are *TI*-frames.

It is easy to see that all *DI*-entailments are *TI*-entailments.

If $\{\gamma_1, \ldots, \gamma_n\} \vDash_{DI} \alpha$, then $\{\gamma_1, \ldots, \gamma_n\} \vDash_{TI} \alpha$.

Since the class of *TI*-frames is a subset of the class of *DI*-frames, any entailment that holds in all *DI*-frames also holds in all *TI*-frames. So the semantical system *DI* is contained in the semantical system *TI*.

Moreover, *TI* is a stronger system than *DI*, in that some *TI*-entailments are not *DI*-entailments, because some *DI*-frames are not *TI*-frames. So *TI* is an extension of *DI*. Specifically,

$\{\Box\alpha\} \nvDash_{DI} \alpha$.

Proof. Let W in a frame Fr contain two worlds, w_1 and w_2, such that Rw_1w_2 and Rw_2w_1. R is therefore serial. Now let $\mathbf{v_I}(\alpha, w_1) = \mathbf{F}$ and $\mathbf{v_I}(\alpha, w_2) = \mathbf{T}$. If follows from **SR**-□ that $\mathbf{v_I}(\Box\alpha, w_1) = \mathbf{T}$, since α is true at all accessible worlds, i.e., at w_2. But we have stipulated that $\mathbf{v_I}(\alpha, w_1) = \mathbf{F}$. So there is an interpretation in a *DI*-frame which blocks the entailment.

w_1	$\rightleftarrows$	w_2
$\Box\alpha$		α
T		**T**
α		
F		

TI has some consequences for iterated modalities. We can assert the following meta-theorems, which follow directly from the characteristic consequences of *TI* and Closure. (The converses do not hold.)

$\{\Box\Box\alpha\} \vDash_{TI} \Box\alpha$, and

$\{\Diamond\alpha\} \vDash_{TI} \Diamond\Diamond\alpha$.

Exercise. Give interpretations to show that neither of the converses hold.

The corresponding results hold for arbitrarily long strings of the same modal operator. This is the first *reduction principle* in the series of progressively stronger systems from *KI* to *S5I*. That is, any sentence beginning with a string of *n* '□' operators entails a sentence with only one, and any sentence beginning with one '◊' operator entails a sentence beginning with any number *n* operators.

In systems weaker than *TI*, a sentence α true at a world may strictly imply (at that world) a sentence β without β's being true there. This limitation of inference for the '⥽' is overcome in *TI*:

$\{\alpha, \alpha \strictif \beta\} \vDash_{TI} \beta$.

Proof. Suppose that for an arbitrary *TI*-frame, world **w** in the frame and interpretation **I** based on the frame, $\mathbf{v_I}(\alpha, \mathbf{w}) = \mathbf{T}$ and $\mathbf{v_I}(\alpha \strictif \beta, \mathbf{w}) = \mathbf{T}$. Then by **SR**-⥽″, at any world $\mathbf{w}_i$ accessible to **w**, if $\mathbf{v_I}(\alpha, \mathbf{w}_i) = \mathbf{T}$, then $\mathbf{v_I}(\beta, \mathbf{w}_i) = \mathbf{T}$. By reflexivity, **w** is one of the $\mathbf{w}_i$s. So if $\mathbf{v_I}(\alpha, \mathbf{w}) = \mathbf{T}$, then $\mathbf{v_I}(\beta, \mathbf{w}) = \mathbf{T}$. Therefore, $\mathbf{v_I}(\beta, \mathbf{w}) = \mathbf{T}$.

The Lewis systems *S2I* and *S3I* also have this feature, which is a consequence of accessibility being reflexive in *S2I*- and *S3I*-frames.

2 The Derivational System *TD*

The derivational system *TD* adds two rules to the derivational rules for *KD*, with the aim of allowing the derivation of α from $\{\Box\alpha\}$. As before, the derivational rule will closely follow the semantical rule. In the semantical system, if a necessity-sentence $\Box\alpha$ is given the value **T** at world **w**, not only is α true at any arbitrary accessible world, but it is true at **w** as well. The easiest way to represent this feature of *TI* is by allowing the elimination of the '□' operator within the current restricted scope line (or where $\Box\alpha$ occurs outside all restricted scope lines).

$\Box$ Elimination

$$\begin{array}{|ll}
\Box\alpha & \\
\vdots & \\
\alpha & \Box\text{ E}
\end{array}$$

We assert without proof that the derivational system *TD* resulting from adding this rule to the derivational rules of *KD* is complete with respect to the semantical system *TI*. We also assert without proof that the derivational system is sound. This claim can be motivated by the way in which a derivation mirrors the semantical reasoning used in the meta-logical derivation above.

To prove: $\{\Box\alpha\} \vdash_{TD} \alpha$

1	$\Box\alpha$	Assumption
2	α	1 $\Box$ E

An alternative rule, for systems with '$\Diamond$' as primitive, would be a rule of $\Diamond$ introduction. We call this a rule of *strong* $\Diamond$ introduction, because it does not require any result within a further restricted scope line.

Strong $\Diamond$ Introduction

$$\begin{array}{|ll}
\alpha & \\
\vdots & \\
\Diamond\alpha & \text{S }\Diamond\text{ I}
\end{array}$$

The rule is sound. If α is true at a world, so is $\Diamond\alpha$, since the world is accessible to itself.

Given a rule of inference which allows the substitution of sentences that are definitionally equivalent by Duality, either rule may be taken as primitive, with the other being a derived rule.

Strong $\Diamond$ Introduction as a derived rule, given Duality

1	α		Assumption
2		$\sim\Diamond\alpha$	Assumption
3		$\Box\sim\alpha$	2 Duality
4		$\sim\alpha$	3 $\Box$ E
5		α	1 Reiteration
6	$\Diamond\alpha$		2-5 $\sim$ E

$\Box$ Elimination as a derived rule, given Duality

1	$\Box\alpha$	Assumption
2	$\quad \sim\alpha$	Assumption
3	$\quad \Box\alpha$	1 Reiteration
4	$\quad \Diamond\sim\alpha$	2 S $\Diamond$ I
5	$\quad \sim\Box\alpha$	4 Duality
6	α	2-5 $\sim$ E

Exercise. Prove that the rule Strong $\Diamond$ Elimination is derived from $\Box$ Elimination, given Duality.

The rules for *TD* allow are strong enough to make the characteristic rules of *DI* derived rules. We will show that Weak $\Diamond$ Elimination is derivable in *TI*.

Weak $\Diamond$ Introduction as a derived rule

$\Box$ $\quad$.	
$\quad$.	
$\quad$.	
$\quad \alpha$	
$\Box\alpha$	$\Box$ I
α	$\Box$ E
$\Diamond\alpha$	S $\Diamond$ I

Exercise. Show that $\sim\Box$ Introduction is derivable in *TI* with Duality.

Because of the semantical result $\{\alpha, \alpha \strictif \beta\} \vDash_{TI} \beta$ which was proved earlier, we can add a derived rule for system *TD* and stronger systems.

$\strictif$ Elimination

α	
$\vdots$	
$\alpha \strictif \beta$	
$\vdots$	
β	$\strictif$ E

Proof of $\strictif$ Elimination as a derived rule is based on the derivation of $\Box(\alpha \supset \beta)$ from $\alpha \strictif \beta$ and the elimination of the '$\Box$' by the *TD* rule of $\Box$ Elimination.

$\strictif$ Elimination as a derived rule

	α	
	$\alpha \strictif \beta$	
$\Box$	$\quad \alpha \supset \beta$	SR-$\strictif$
	$\Box(\alpha \supset \beta)$	$\Box$ I
	$\alpha \supset \beta$	$\Box$ E
	β	$\supset$ E

3 The Axiom System *T*

The axiom system *T* is obtained by adding to the axioms of *K* the further axiom schema:

$\vdash_T \Box\alpha \supset \alpha.$

This axiom is clearly valid in *TI*. By Bivalence, for any frame **Fr**, any world **w** in **Fr**, and any interpretation **I** based on **Fr**, either $v_{\mathbf{I}}(\Box\alpha, \mathbf{w}) = \mathbf{T}$, or $v_{\mathbf{I}}(\Box\alpha, \mathbf{w}) = \mathbf{F}$. If the latter, then by **SR**-$\supset$, $v_{\mathbf{I}}(\Box\alpha \supset \alpha, \mathbf{w}) = \mathbf{T}$. If the former, then by reflexivity, $v_{\mathbf{I}}(\alpha, \mathbf{w}) = \mathbf{T}$, in which case by **SR**-$\supset$, $v_{\mathbf{I}}(\Box\alpha \supset \alpha, \mathbf{w}) = \mathbf{T}$. So in both possible cases, $v_{\mathbf{I}}(\Box\alpha \supset \alpha, \mathbf{w}) = \mathbf{T}$, which was to be proved.

Exercise. Prove that the *T*-theorem $\alpha \supset \Diamond\alpha$ is valid in *TI*.

4 Applications of the *T*-Systems

4.1 Alethic Modal Logic

The *T*-systems are more adequate than are the weaker *D*-systems for representing consequence relations involving logical possibility and necessity. As noted in a previous module, we want to be able to say that what is logically necessary is true, and that what is true is logically possible. Both these *desiderata* are satisfied in the *T*-systems.

With hypothetical necessity, the *T*-systems seem to have no clear-cut advantage over weaker systems. Given that $\Box\alpha$ holds at a world, we might think that the condition governing the necessity holds at that world, and hence that α is true there. For example, suppose that the '$\Box$' operator is meant to signify physical necessity, where the condition is that a set of laws of nature hold. If it is true in a given world that those laws hold, it seems that they should apply to that world.

Now we might want to consider worlds which are subject to laws different from the laws governing the world in question. Then it seems right to say that when those laws are expressed by '$\Box$,' the truth-value of the sentence $\Box\alpha$ should depend only on the value of α at the worlds where the condition holds, but not necessarily at the world where the laws do not hold.

4.2 Conditional Logic

The *T*-systems overcome a problem for the logic of strict implication. It was noted in an earlier module that in the semantical rules for *KI*, there are frames with worlds where α and $\alpha \strictif \beta$ are true, but β is false. (*DI* is no better off in this respect.) These are precisely worlds which are not accessible to themselves. Imposing the reflexivity requirement on accessibility in *TI* now allows the entailment $\{\alpha, \alpha \strictif \beta\} \vDash_{TI} \beta$.

If the fish-hook is understood as indicating a relation of implication given that a certain condition holds, then the reasoning just used in the case of hypothetical necessity applies. Suppose we take $\alpha \multimap \beta$ to indicate that at all worlds where a condition holds, if α is true, then β is true. If that sentence holds at the home world, then it is natural to take it that the condition holds there as well. Then if α is true there, β should also be true there. So, the only time the T-systems might be considered too strong for this application is when the fish-hook is intended to express a condition that does not hold in the home world. It is hard to say what such an application would be.

4.3 Deontic Logic

The T-systems are too strong for deontic logic, given standard views about obligation and permission. With respect to morality and law, what is obligatory is in many cases not what actually holds, and what actually holds is often not permitted. And even if in fact the real were to conform to the ideal, we would not want to say this is so as a matter of the logic of obligation.

4.4 Doxastic Logic

The T-systems are also too strong for doxastic logic. Even if it is a matter of the logic that all valid sentences are believed, it is not a matter of logic that every belief, no matter what its content, is true.

4.5 Epistemic Logic

Most philosophers believe that it is part of the notion of knowledge that what is known is true, so the characteristic consequence in T, $\{\mathrm{K}_{s,t}\alpha\} \vDash_{TI} \alpha$, should hold. Similarily if α is the case, then it ought to be compatible with what I know, and this conforms to the fact that $\{\alpha\} \vDash_{TI} \mathrm{F}_{s,t}\alpha$. Finally, there is a consequence-relation holding in T that is relevant to epistemic logic:

$\{\mathrm{K}_{s,t}\alpha\} \vDash_{TI} \mathrm{F}_{s,t}\mathrm{K}_{s,t}\alpha$.

If a subject knows that α, then for all he knows, he knows that α. This is an immediate result of the use of the epistemic correlate of Strong $\Diamond$ Introduction. On the semantic side, the consequence holds because the world at which the subject knows is an accessible world to itself, and accessible worlds express what is compatible with what a subject knows. So the fact that the subject has knowledge that α is compatible with what the subject knows. An equivalent result is $\{\mathrm{K}_{s,t}\alpha\} \vDash_{TI} \sim\mathrm{K}_{s,t}\sim\mathrm{K}_{s,t}\alpha$. It seems that my knowing that α implies that I do not know that I do not know that α.

4.6 Temporal Logic

A temporal logic of the past and future is not appropriately based on T. What is the case at all future times does not have to be the case now, though it may be. Nor is what is the case now such that it is the case at some future times. Similar remarks hold for what has always been the case and what once was the case, respectively.

It should be noted that there are logics in which ‘F’ indicates what is the case now or at some future time, and ‘G’ indicates what is and always will be the case. Corresponding remarks hold for the ‘P’ and ‘H’ operators, respectively. For such logics, the characteristic consequence in the T-systems seems appropriate.