Module 11
S5 and Equivalent Systems

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Contents

The *S5*-systems are the strongest normal systems. That is, they are the strongest modal systems in which the semantical rules behave uniformly at all the worlds in a frame. In one formulation of the semantics, in every frame, all worlds must be accessible to all worlds. No stronger constraint can be placed on the accessibility relation.[1]

The *S5*-systems are extensions of the *B*-systems and the *S4*-systems, and therefore are extensions of the *K*-, *D*-, and *T*-systems as well. As with the other families of systems, we will discuss the semantical system first, then the derivational system, then, briefly, the axiom system, and finally applications.

[1]Axiom systems known as *S6*, *S7*, *S8*, and *S9* are neither contained in nor contained in *S5*, but are formed by adding axioms to *S2* or systems based on *S2*.

1 The Semantical System *S5I*

S5I is unusual in that it allows for a number of different formulations, each yielding the same entailments. In this section, we will examine four ways of formulating the semantics.

1.1 Accessibility as Reflexive and Euclidean

One way to build *S5I* is to begin with the semantical systems *TI*, in which accessibility is reflexive, and add the further requirement that accessibility be *euclidean*.

$\mathbf{R}$ is euclidean iff $(\Pi\mathbf{x})(\Pi\mathbf{y})(\Pi\mathbf{z})((\mathbf{Rxy} \wedge \mathbf{Rxz}) \rightarrow \mathbf{Ryz})$.

If a world is accessible to any two (not necessarily distinct) worlds, then those two worlds are accessible to each other. (The definition of a euclidean relation specifies that one is accessible to the other. But each of the two accessible worlds could serve as the value of '**y**' or of '**z**,' so they are mutually accessible.)

Applied to frames, this means that if a world $\mathbf{w}_i$ is accessible to $\mathbf{w}$, and a world $\mathbf{w}_j$ is accessible to $\mathbf{w}$ then $\mathbf{w}_j$ is accessible to $\mathbf{w}_i$. We can define an *S5I*-frame as a set $\langle \mathbf{W}, \mathbf{R} \rangle$, such that :

$(\Pi\mathbf{w})(\mathbf{w} \in \mathbf{W} \rightarrow \mathbf{Rww})$, and

$(\Pi\mathbf{w})(\Pi\mathbf{w}_i)(\Pi\mathbf{w}_j)((((\mathbf{w} \in \mathbf{W} \bigwedge \mathbf{w}_i \in \mathbf{W}) \wedge \mathbf{w}_j \in \mathbf{W}) \bigwedge(\mathbf{Rww}_i \wedge \mathbf{Rww}_j)) \rightarrow \mathbf{Rw}_i\mathbf{w}_j)$.

The euclidean character of the accessibility relation in *S5I*-frames yields what will be called the characteristic consequence of the *S5*-systems.

$\{\Diamond\alpha\} \vDash_{S5I} \Box\Diamond\alpha$.

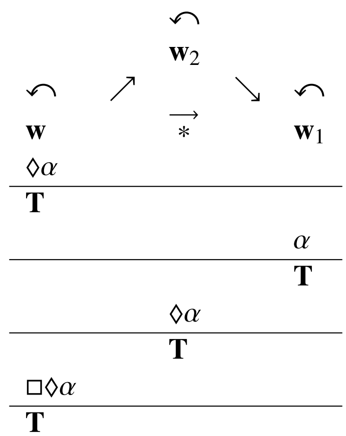
The proof of the entailment can be given using a meta-logical derivation.

Semantical proof that: $\{\Diamond\alpha\} \vDash_{S5I} \Box\Diamond\alpha$

1	$\mathbf{v_I}(\Diamond\alpha, \mathbf{w}) = \mathbf{T}$	Assumption
2	$(\mathbf{Rww}_2 \bigwedge \mathbf{Rww}_1) \rightarrow \mathbf{Rw}_2\mathbf{w}_1$	**R** is euclidean
3	$(\Sigma\mathbf{w}_i)(\mathbf{Rww}_i \bigwedge \mathbf{v_I}(\alpha, \mathbf{w}_i) = \mathbf{T})$	**SR**-$\Diamond$
4	$\quad \mathbf{Rww}_1 \bigwedge \mathbf{v_I}(\alpha, \mathbf{w}_1) = \mathbf{T}$	Assumption
5	$\quad\quad \mathbf{Rww}_2$	Assumption
6	$\quad\quad \mathbf{Rww}_1$	4 $\bigwedge$ E
7	$\quad\quad \mathbf{Rww}_2 \bigwedge \mathbf{Rww}_1$	5 6 $\bigwedge$ I
8	$\quad\quad \mathbf{Rw}_2\mathbf{w}_1$	2 7 $\rightarrow$ E
9	$\quad\quad \mathbf{v_I}(\alpha, \mathbf{w}_1) = \mathbf{T}$	4 $\bigwedge$ E
10	$\quad\quad \mathbf{Rw}_2\mathbf{w}_1 \bigwedge \mathbf{v_I}(\alpha, \mathbf{w}_1) = \mathbf{T}$	8 9 $\bigwedge$ I
11	$\quad\quad (\Sigma\mathbf{w}_i)(\mathbf{Rw}_2\mathbf{w}_i \bigwedge \mathbf{v_I}(\alpha, \mathbf{w}_i) = \mathbf{T})$	10 Σ I
12	$\quad\quad \mathbf{v_I}(\Diamond\alpha, \mathbf{w}_2) = \mathbf{T}$	11 **SR**-$\Diamond$
13	$\quad \mathbf{Rww}_2 \rightarrow \mathbf{v_I}(\Diamond\alpha, \mathbf{w}_2) = \mathbf{T}$	5-12 $\rightarrow$ I
14	$\quad \mathbf{v_I}(\Box\Diamond\alpha, \mathbf{w}) = \mathbf{T}$	13 **SR**-$\Box$
15	$\mathbf{v_I}(\Box\Diamond\alpha, \mathbf{w}) = \mathbf{T}$	3 4-14 Σ E

Note that this derivation does not depend on any restrictions on **R** other than its being euclidean. So being euclidean could be added to the semantical systems *KI* or *DI*, rather than *TI*, to produce other semantical systems with the charateristic *S5* consequence relation.

The reasoning can be represented graphically as follows.

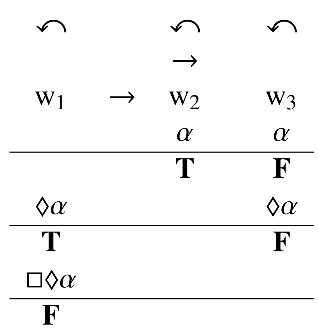


Note that only the requirement that **R** be euclidean, and not that it be reflexive, plays a role in this proof.

Since **R** is reflexive in an *S5I*-frame, *S5I* contains *TI*. But the characteristic *S5I* entailment does not hold in *TI*.

$\{\Diamond\alpha\} \nvDash_{TI} \Box\Diamond\alpha$.

Proof. Let W in a frame Fr contain three worlds, w_1, w_2, and w_3, such that Rw_1w_1, Rw_2w_2, Rw_3w_3, Rw_1w_2, and Rw_1w_3. R is therefore reflexive, and so Fr is a *TI*-frame. Now let $\mathbf{v_I}(\alpha, w_2) = \mathbf{T}$ and $\mathbf{v_I}(\alpha, w_3) = \mathbf{F}$. If follows from **SR**-◊ that $\mathbf{v_I}(\Diamond\alpha, w_1) = \mathbf{T}$ and $\mathbf{v_I}(\Diamond\alpha, w_3) = \mathbf{F}$. From the second result it follows from **SR**-□ that $\mathbf{v_I}(\Box\Diamond\alpha, w_1) = \mathbf{F}$.



It can also be proved that *S5I* contains both *S4I* and *BI*, and that it is stronger than each of the two systems. To prove the former claim, we show that if a relation is both reflexive and euclidean, it is also symmetrical and transitive, and hence that any *S4I*-frame and any *BI*-frame is an *S5I*-frame.[2]

[2]If *n* quantifier introductions or eliminations are used in a single step, the rule will be cited as having been used *n* times, as in '2 Π E x 3.'

Sketch of a proof that if R is reflexive and euclidean, then R is symmetrical

1	$(\Pi\mathbf{x})\mathbf{Rxx}$	Assumption
2	$(\Pi\mathbf{x})(\Pi\mathbf{y})(\Pi\mathbf{z})((\mathbf{Rxy} \wedge \mathbf{Rxz})) \rightarrow \mathbf{Ryz})$	Assumption
3	$\mathbf{Rab}$	Assumption
4	$\mathbf{Raa}$	1 Π E
5	$(\mathbf{Rab} \bigwedge \mathbf{Raa}) \rightarrow \mathbf{Rba}$	2 Π E x 3
6	$\mathbf{Rab} \bigwedge \mathbf{Raa}$	3 5 ⋀ I
7	$\mathbf{Rba}$	5 6 → E
8	$\mathbf{Rab} \rightarrow \mathbf{Rba}$	3-7 → I
9	$(\Pi\mathbf{x})(\Pi\mathbf{y})(\mathbf{Rxy} \rightarrow \mathbf{Ryx})$	8 Π I x 2

Sketch of a proof that if R is reflexive and euclidean, then R is transitive

1	$(\Pi\mathbf{x})\mathbf{Rxx}$	Assumption
2	$(\Pi\mathbf{x})(\Pi\mathbf{y})(\Pi\mathbf{z})((\mathbf{Rxy} \wedge \mathbf{Rxz})) \rightarrow \mathbf{Ryz})$	Assumption
3	$\mathbf{Rab} \bigwedge \mathbf{Rbc}$	Assumption
4	$\mathbf{Raa}$	1 Π E
5	$(\mathbf{Rab} \bigwedge \mathbf{Raa}) \rightarrow \mathbf{Rba}$	2 Π E x 3
6	$\mathbf{Rab}$	3 ⋀ E
7	$\mathbf{Rab} \bigwedge \mathbf{Raa}$	4 6 ⋀ I
8	$\mathbf{Rba}$	5 7 → E
9	$\mathbf{Rbc}$	3 ⋀ E
10	$(\mathbf{Rba} \bigwedge \mathbf{Rbc}) \rightarrow \mathbf{Rac}$	2 Π E x 3
11	$\mathbf{Rba} \bigwedge \mathbf{Rbc}$	8 9 ⋀ I
12	$\mathbf{Rac}$	10 11 → E
13	$(\mathbf{Rab} \bigwedge \mathbf{Rbc}) \rightarrow \mathbf{Rac}$	3-12 → I
14	$(\Pi\mathbf{x})(\Pi\mathbf{y})(\Pi\mathbf{z})((\mathbf{Rxy} \bigwedge \mathbf{Ryz}) \rightarrow \mathbf{Rxz})$	13 Π I x 3

It can be proved that the characteristic *S5I* entailment fails in both *S4I* and *BI*.

$\{\Diamond\alpha\} \nvDash_{S4I} \Box\Diamond\alpha$.

Proof. Let W in a frame Fr contain three worlds, w_1, w_2, and w_3, such that Rw_1w_1, Rw_2w_2, Rw_3w_3, Rw_1w_2, and Rw_1w_3. R is therefore reflexive and (trivially) transitive, so Fr is a *S4I*-frame.[3] Now let $\mathbf{v_I}(\alpha, w_2) = \mathbf{T}$ and $\mathbf{v_I}(\alpha, w_3) = \mathbf{F}$. It follows from **SR**-◊ that $\mathbf{v_I}(\Diamond\alpha, w_1) = \mathbf{T}$. From the same semantical rule it follows that $\mathbf{v_I}(\Diamond\alpha, w_3) = \mathbf{F}$. Therefore, by **SR**-□, $\mathbf{v_I}(\Box\Diamond\alpha, w_1) = \mathbf{F}$.

[3] R is trivially transitive because no two distinct worlds meet the antecedent of the conditional defining transitivity.

$\curvearrowleft$		$\curvearrowleft$	$\curvearrowleft$
		$\rightarrow$	
w_1	$\rightarrow$	w_2	w_3
		α	α
		T	**F**
$\Diamond\alpha$			$\Diamond\alpha$
T			**F**
$\Box\Diamond\alpha$			
F			

$\{\Diamond\alpha\} \nvDash_{BI} \Box\Diamond\alpha$.

Proof. Let W in a frame Fr contain two worlds, w_1, w_2, such that Rw_1w_1, Rw_2w_2, Rw_1w_2, and Rw_2w_1. R is therefore reflexive and symmetric, so Fr is a *BI*-frame. Now let $\mathbf{v_I}(\alpha, w_1) = \mathbf{F}$, and $\mathbf{v_I}(\alpha, w_2) = \mathbf{T}$. It follows from **SR**-$\Diamond$ that $\mathbf{v_I}(\Diamond\alpha, w_1) = \mathbf{T}$. From the same semantical rule it follows that $\mathbf{v_I}(\Diamond\alpha, w_1) = \mathbf{F}$, since α is false at both w_1. Therefore, by **SR**-$\Box$, $\mathbf{v_I}(\Box\Diamond\alpha, w_1) = \mathbf{F}$.

$\curvearrowleft$		$\curvearrowleft$
w_1	$\leftrightarrows$	w_2
α		α
F		**T**
$\Diamond\alpha$		
T		
$\Box\Diamond\alpha$		
F		

1.2 Accessibility as Reflexive, Transitive, and Symmetrical

A second way to generate the semantical system is to require that an *S5I*-frame is both transitive and symmetrical as well as being reflexive. In this way, *S5I* is built on both **S4I** and **BI**. We can define an *S5I*-frame as a set $\langle \mathbf{W}, \mathbf{R} \rangle$, such that :

$(\Pi\mathbf{w})(\mathbf{w} \in \mathbf{W} \rightarrow \mathbf{Rww})$, and

$(\Pi\mathbf{w})(\Pi\mathbf{w}_i)(\Pi\mathbf{w}_j)((((\mathbf{w} \in \mathbf{W} \bigwedge \mathbf{w}_i \in \mathbf{W}) \bigwedge \mathbf{w}_j \in \mathbf{W}) \bigwedge(\mathbf{Rww}_i \wedge \mathbf{Rww}_i)) \rightarrow \mathbf{Rw}_i\mathbf{w}_j)$, and

$(\Pi\mathbf{w})(\Pi\mathbf{w}_i)(((\mathbf{w} \in \mathbf{W} \bigwedge \mathbf{w}_i \in \mathbf{W}) \bigwedge \mathbf{Rww}_i) \rightarrow \mathbf{Rw}_i\mathbf{w})$.

We have already seen that if a relation is euclidean, then it is transitive and symmetrical. We can also prove the converse.

Sketch of a proof that if R is transitive and symmetrical, then it is euclidean

1	$(\Pi\mathbf{x})(\Pi\mathbf{y})(\Pi\mathbf{z})((\mathbf{Rxy} \wedge \mathbf{Ryz})) \rightarrow \mathbf{Rxz})$	Assumption
2	$(\Pi\mathbf{x})(\Pi\mathbf{y})(\mathbf{Rxy} \rightarrow \mathbf{Ryx})$	Assumption
3	$\mathbf{Rab} \bigwedge \mathbf{Rac}$	Assumption
4	$(\mathbf{Rba} \bigwedge \mathbf{Rac}) \rightarrow \mathbf{Rbc}$	1 Π E x 3
5	$\mathbf{Rab} \rightarrow \mathbf{Rba}$	2 Π E x 2
6	$\mathbf{Rab}$	3 ⋀ E
7	$\mathbf{Rba}$	4 6 → E
8	$\mathbf{Rac}$	3 ⋀ E
9	$\mathbf{Rba} \bigwedge \mathbf{Rac}$	7 8 ⋀ I
10	$\mathbf{Rbc}$	4 9 → E
11	$(\mathbf{Rab} \bigwedge \mathbf{Rac}) \rightarrow \mathbf{Rbc}$	3-10 → I
12	$(\Pi\mathbf{x})(\Pi\mathbf{y})(\Pi\mathbf{z})((\mathbf{Rxy} \bigwedge \mathbf{Rxz}) \rightarrow \mathbf{Ryz})$	11 Π I x 3

From this result it follows that the first condition on *S5I*-frames, that they be euclidean, is a consequence of the two conditions just stated. Since being euclidean is equivalent to being transitive and symmetrical, the semantical system would produce the same results if it is structured in either of the two ways. An example is a proof of the characteristic entailment.

Sketch of a semantical proof that: $\{\Diamond\alpha\} \vDash_{S5I} \Box\Diamond\alpha$

1	$\mathbf{v_I}(\Diamond\alpha, \mathbf{w}) = \mathbf{T}$	Assumption
2	$(\mathbf{Rw}_2\mathbf{w} \wedge \mathbf{Rww}_1) \rightarrow \mathbf{Rw}_2\mathbf{w}_1$	**R** is transitive
3	$\mathbf{Rww}_2 \rightarrow \mathbf{Rw}_2\mathbf{w}$	**R** is symmetrical
4	$(\Sigma\mathbf{w}_i)(\mathbf{Rww}_i \wedge \mathbf{v_I}(\alpha, \mathbf{w}_i) = \mathbf{T})$	**SR**-◊C
5	$\mathbf{Rww}_1 \wedge \mathbf{v_I}(\alpha, \mathbf{w}_1) = \mathbf{T}$	Assumption
6	$\mathbf{Rww}_2$	Assumption
7	$\mathbf{Rw}_2\mathbf{w}$	3 6 → E
8	$\mathbf{Rww}_1$	5 ⋀ E
9	$\mathbf{Rw}_2\mathbf{w} \wedge \mathbf{Rww}_1$	7 8 ⋀ I
10	$\mathbf{Rw}_2\mathbf{w}_1$	3 9 → E
11	$\mathbf{v_I}(\alpha, \mathbf{w}_1) = \mathbf{T}$	5 ⋀ E
12	$\mathbf{Rw}_2\mathbf{w}_1 \wedge \mathbf{v_I}(\alpha, \mathbf{w}_1) = \mathbf{T}$	10 11 ⋀ I
13	$(\Sigma\mathbf{w}_i)(\mathbf{Rw}_2\mathbf{w}_i \wedge \mathbf{v_I}(\alpha, \mathbf{w}_i) = \mathbf{T})$	12 Σ I
14	$\mathbf{v_I}(\Diamond\alpha, \mathbf{w}_2) = \mathbf{T}$	13 **SR**-◊
15	$\mathbf{Rww}_2 \rightarrow \mathbf{v_I}(\Diamond\alpha, \mathbf{w}_2) = \mathbf{T}$	5-14 → I
16	$\mathbf{v_I}(\Box\Diamond\alpha, \mathbf{w}) = \mathbf{T}$	15 **SR**-□
17	$\mathbf{v_I}(\Box\Diamond\alpha, \mathbf{w}) = \mathbf{T}$	4 5-16 Σ E

1.3 Accessibility as an Equivalence Relation

A relation that is reflexive, transitive, and symmetrical is called an *equivalence relation*. So, we can define the accessibility relation for *S5I* simply as being an equivalence relation. We can define an *S5I*-frame as a set $\langle \mathbf{W}, \mathbf{R} \rangle$, such that :

R is an equivalence relation.

Examples of equivalence relations are *being identical to*, *being the same size as*, *being semantically equivalent to*. In the case of semantical equivalence, we can say that every sentence in a language, say *SL*, is semantically equivalent to itself. If α is semantically equivalent to β, and β is semantically equivalent to γ, then α is semantically equivalent to γ. And if α is semantically equivalent to β, then β is semantically equivalent to α.

Note that in *SI*, not every sentence is equivalent to every other sentence. Instead, sentences which stand in the equivalence relation to one another constitute an *equivalence class* of sentences. For example, the class of all valid sentences is an equivalence class, as is the class of all negations of valid sentences.[4] In the case of being the same size as, for each size there is an equivalence class of things of that size.

Applied to frames, this means that **W** can be divided into equivalence classes, such that the members of each class stand in the equivalence relation to themselves and all the other members of the class. It will

[4]The negations of valid sentences are often called "contradictions," or "inconsistent" or "logically false" sentences. So the class of all "contradictions" is an equivalence class.

be shown that no member $\mathbf{w}_i$ of one equivalence class is accessible to or from any member $\mathbf{w}_j$ of any other equivalence class.

If it were the case that $\mathbf{w}_i$ is accessible to $\mathbf{w}_j$, then $\mathbf{w}_i$ would stand in an equivalence relation to $\mathbf{w}_j$, and by the properties of the equivalence relation, it would stand in an equivalence relation to all the other members of the class. Moreover, because $\mathbf{w}_i$ stands in an equivalence relation to all the members of *its own* equivalence class, all of them would stand in the equivalence relation to all the members of the other equivalence class. Then there would only be one equivalence class, which is contrary to the assumption that there are two equivalence classes. We can re-prove the characteristic entailment of *S5I* in terms of the equivalence relation. Suppose that $\mathbf{v_I}(\Diamond\alpha, \mathbf{w}) = \mathbf{T}$. Then there is some world $\mathbf{w}_i$ such that $\mathbf{Rww}_i$ where $\mathbf{v_I}(\alpha, \mathbf{w}_i) = \mathbf{T}$. Because $\mathbf{R}$ is an equivalence class, all worlds $\mathbf{w}_j$ that are accessible to $\mathbf{w}$ are such that $\mathbf{w}_i$ is accessible to them. Therefore at all $\mathbf{w}_j$ accessible to $\mathbf{w}$, $\mathbf{v_I}(\Diamond\alpha, \mathbf{w}_j) = \mathbf{T}$, in which case $\mathbf{v_I}(\Box\Diamond\alpha, \mathbf{w}) = \mathbf{T}$.

1.4 Accessibility as a Universal Relation

If we take accessibility to be an equivalence relation, the set of worlds $\mathbf{W}$ might be partitioned into a number of isolated equivalence classes. That this kind of frame is permitted does not affect any of the semantical results we have obtained. From the standpoint of a given equivalence class, the truth-value assigments at worlds belonging to the other classes are completely irrelevant.

If, for example, $\Diamond\alpha$ is true at a world $\mathbf{w}$, then the truth-value of $\Box\Diamond\alpha$ will be determined by the value of $\Diamond\alpha$ at all worlds accessible to $\mathbf{w}$, which is to say all worlds in the equivalence class. The validity in *S5I* of $\Diamond\alpha \supset \Box\Diamond\alpha$ depends on its being true in all worlds on all interpretations given any *S5I*-frame. And it is true at each world in each equivalence class, so it is true at all the worlds in a frame. So there is no result that holds for a partitioned set of worlds that does not hold for an unpartitioned set, where all the worlds in a frame stand in the equivalence relation.

This suggests that we can get the same semantical results for *S5I* if we make accessibility a *universal* relation, i.e., a relation holding between each world in $\mathbf{W}$. In general:

$\mathbf{R}$ is **universal** if and only if $(\Pi\mathbf{x})(\Pi\mathbf{y})\mathbf{Rxy}$.

We can define an *S5I*-frame as a set $\langle \mathbf{W}, \mathbf{R} \rangle$, such that :

$(\Pi\mathbf{w})(\Pi\mathbf{w}_i)((\mathbf{w} \in \mathbf{W} \bigwedge \mathbf{w}_i \in \mathbf{W}) \rightarrow \mathbf{Rww}_i)$.

This simplifies considerably the semantical proofs in *S5I*. For example, suppose that on an *S5I*-frame **Fr** of this type, on an arbitrary interpretation *I* based on **Fr**, and at an arbitrary world $\mathbf{w}$ in **Fr**, that $\mathbf{v_I}(\Diamond\alpha, \mathbf{w}) = \mathbf{T}$. Then at some accessible world $\mathbf{w}_i$, α is true. Because $\mathbf{R}$ is universal, $\mathbf{w}_i$ is accessible to all worlds, so that $\Diamond\alpha$ is true at all worlds. And because $\Diamond\alpha$ is true at all worlds, it is true at all worlds accessible to $\mathbf{w}$, in which case $\Box\Diamond\alpha$ is true at $\mathbf{w}$.

1.5 Semantics without the Accessibility Relation

As the preceding proof indicates, when accessibility is a universal relation, it plays no role in determining semantical results. We could just as well have reasoned as follows. Suppose that at an arbitrary world $\mathbf{w}$, $\mathbf{v_I}(\Diamond\alpha,\mathbf{w}) = \mathbf{T}$. Then at some world $\mathbf{w}_i$, $\mathbf{v_I}(\alpha, \mathbf{w}_i) = \mathbf{T}$. So $\mathbf{v_I}(\Diamond\alpha, \mathbf{w}_j) = \mathbf{T}$ at all worlds $\mathbf{w}_j$, in which case $\mathbf{v_I}(\Box\Diamond\alpha, \mathbf{w}) = \mathbf{T}$.

So yet another version of the semantical system *SI5* is one in which there is no relation of accessibility. Then a frame would be only an ordered single consisting of a set of worlds $\langle \mathbf{W} \rangle$, and an interpretation would be a pair $\langle \mathbf{W}, \mathbf{v} \rangle$.

If this simplifying move is to be made, the semantical rules must be changed to omit the reference to accessibility. The following are the semantical rules for the two main modal operators, '□' and '◊.'

SR-□(5) $\mathbf{v_I}(\Box\alpha, \mathbf{w})$=**T** if and only if $\mathbf{v_I}(\alpha, \mathbf{w}_i)$=**T** at all worlds $\mathbf{w}_i$ in **I**; $\mathbf{v_I}(\Box\alpha, \mathbf{w})$=**F** if and only if $\mathbf{v_I}(\alpha, \mathbf{w}_i)$=**F** at some world $\mathbf{w}_i$ in **I**;.

SR-◊(5) $\mathbf{v_I}(\Diamond\alpha, \mathbf{w})$=**T** if and only if $\mathbf{v_I}(\alpha, \mathbf{w}_i)$=**T** at some world $\mathbf{w}_i$ in **I**; $\mathbf{v_I}(\Diamond\alpha, \mathbf{w})$=**F** if and only if $\mathbf{v_I}(\alpha, \mathbf{w}_i)$=**F** at all worlds $\mathbf{w}_i$ in **I**.

It is easily proved that these semantical rules are equivalent to the semantical rules with the accessibility relation **R** if **R** is a universal relation.

If **R** is universal, then reference to **R** in **SR**-□ can be eliminated.

1	$(\Pi\mathbf{w})(\Pi\mathbf{w}_i)(\mathbf{Rww}_i)$	**R** is universal
2	$(\Pi\mathbf{w}_i)(\mathbf{Rww}_i \rightarrow \mathbf{v_I}(\alpha, \mathbf{w}_i) = \mathbf{T})$	**SR**-□
3	$(\Pi\mathbf{w}_i)(\mathbf{Rww}_i)$	1 Π E
4	$\mathbf{Rww}_1$	3 Π E
5	$\mathbf{Rww}_1 \rightarrow \mathbf{v_I}(\alpha, \mathbf{w}_1) = \mathbf{T}$	2 Π E
6	$\mathbf{v_I}(\alpha, \mathbf{w}_1) = \mathbf{T}$	4 5 → E
7	$(\Pi\mathbf{w})\mathbf{v_I}(\alpha, \mathbf{w}) = \mathbf{T}$	6 Π I

Exercise. Prove the converse.

1.6 Semantics without Possible Worlds

A final simplification generates what is roughly the semantical system of Carnap, 1946.[5] It is possible to dispense with possible worlds altogether in favor of, in effect, possible interpretations.

An interpretation is defined as in *SI*, the semantical system for Sentence Logic. It consists of a one-place valuation function which assigns the value **T** or the value **F**, but not both, to each sentence letter. The semantical rules for the non-modal operators are just as they are with *SI*.

Carnap's idea for the modal operator '□' was that it should reflect the semantical concept of "L-truth." His informal definition of L-truth (in a system S) is truth "in such a way as . . . can be established on the basis of the semantical rules of the system S alone, without any reference to (extra-linguistic) facts."[6] The formal definition of L-truth (in terms of the semantical system *SI* presented here) is truth in every interpretation, i.e., validity.

The semantical rule which is supposed to reflect the notion of L-truth is unusual. The two possible outcomes of the use of the rule are not truth and falsehood within a single interpretation, as with *SI*. Instead, the alternatives are truth in *all* interpretations or falsehood in *all* interpretations:

$\mathbf{v_I}(\Box\alpha)$=**T** if and only if $(\Pi\mathbf{I})\mathbf{v_I}(\alpha)$=**T**; $\mathbf{v_I}(\Box\alpha)$=**F** if and only if $(\Sigma\mathbf{I})\mathbf{v_I}(\alpha)$=**F**.

The corresponding rule for possibility then would be as follows:

$\mathbf{v_I}(\Diamond\alpha)$=**T** if and only if $(\Sigma\mathbf{I})\mathbf{v_I}(\alpha)$=**T**; ($\mathbf{v_I}(\Diamond\alpha)$=**F** if and only if $(\Pi\mathbf{I})\mathbf{v_I}(\alpha)$=**F**.

[5]"Modalities and Quantification," *Journal of Symbolic Logic* 11 (1946), pp. 33-64, and *Meaning and Necessity* (1948). The chief differences are that Carnap does not use the modern notion of an interpretation, in which a valuation function assigns truth-values to sentences. In place of interpretations he uses "state-descriptions," which in Sentential Logic would be a set of sentence letters. What corresponds to the assignment of '**T**' is membership in the set, and what corresponds to the assignment of '**F**' is non-membership in the set. Hintikka based his early modal semantics on this method, but Kripke's formulation in terms of valuation functions ultimately became the standard.

[6]*Meaning and Necessity*, Second Edition, p. 10.

A consequence of these rules is that the truth-value of any modal sentence is the same on all interpretations. All necessity-sentences true on a given interpretation are true on all interpretations, and all possiblity-sentences true on a given interpretation are true on all interpretations. Suppose that on a given interpretation **I**, $\mathbf{v_I}(\Box\alpha)$=**T**. Then by the semantical rule, $(\Pi\mathbf{I})\mathbf{v_I}(\alpha)$=**T**. Using the semantical rule in the other direction, we can assert that $\Box\alpha$ has the value **T** on an arbitrary interpretation, and so it has the value **T** on all interpretations.

Exercise. Prove that that the corresponding claim holds for possibility.

The following is a proof of the characteristic *S5I* entailment. In giving the proof, we will appeal to the following lemma, which states the result just given, for the case of the '$\Diamond$':

$\mathbf{v_I}(\Diamond\alpha)=\mathbf{T} \rightarrow (\Pi\mathbf{I})(\mathbf{v_I}(\Diamond\alpha)=\mathbf{T})$

Now it is easy to prove the entailment.

Sketch of a semantical proof that: $\{\Diamond\alpha\} \vDash_{S5I} \Box\Diamond\alpha$

1	$\mathbf{v_I}(\Diamond\alpha) = \mathbf{T}$	Assumption
2	$(\Pi\mathbf{I})(\mathbf{v_I}(\Diamond\alpha) = \mathbf{T})$	Lemma
3	$\mathbf{v_I}(\Box\Diamond\alpha) = \mathbf{T}$	2 Rule for '$\Box$'

Carnap had good reason to give the semantical rules he did, as his goal was to represent logical necessity. But given the structure of the rules he adopted to represent logical necessity, there is no hint of any kind of relativity of the truth-values of modal sentences. It is small wonder that Carnap did not find in this semantical system a key to possible-worlds semantics with the accessibility relation.

1.7 Reduction of Modalities in *S5I*

We saw that in the *S4*-systems, there remain a number of irreducible strings of modalities. That is, there are sentences which are prefixed with a string of two or more modal operators (of both kinds) which are not equivalent to sentences with fewer operators. In the *S5*-systems, this result does not hold. Every sentence prefixed with a string of two or more modal operators is equivalent to a sentence with only one operator, the innermost one. This is the ultimate "reduction" of modalities. The only further reduction would be to allow a sentence with a single modal operator as its main operator to be equivalent to a sentence with no moral operator. This would collapse the modal system into the non-modal system.

The modal sentences $\Diamond\Box\Diamond\alpha$ and $\Box\Diamond\alpha$, are reducible to $\Diamond\alpha$, and the modal sentences $\Box\Diamond\Box\alpha$ and $\Diamond\Box\alpha$, are reducible to the $\Box\alpha$. We have already seen how $\Box\Diamond\alpha$ reduces to $\Diamond\alpha$. The result from left to right holds in the *TI* and the result from right to left has been shown in this module to hold in *S5I* (in its various guises).

Exercise. Prove the other three reductions in the variants of the semantical system.

2 The Derivational System *S5D*

As was the case with the symmetrical accessibility relation in the semantical system *BI*, the euclidean relation is not readily mapped onto the structure of Fitch-style derivations. So as with *BD*, we shall resort to a device that simulates the effect of a euclidean accessibility relation. Inspection of the characteristic *S5* entailment, $\{\Diamond\alpha\} \vDash_{S5I} \Box\Diamond\alpha$, shows that if $\Diamond\alpha$ is taken to be true at **w**, then it must also be true at all accessible worlds, and hence at an arbitrary accessible world. So a sound rule would be one that allows us to use Strict Reiteration to bring a sentence of the form $\Diamond\alpha$ across a restricted scope line intact.

Strict Reiteration for '□' (5)

	$\Diamond\alpha$	Already derived
□	$\vdots$	
	$\Diamond\alpha$	SR-□(5)

We shall assume that the derivational system resulting from the addition of SR(5) to the rules for *TD* is complete as well as being sound.

With SR(5), proof of the characteristic derivational-relation is straightforward.

To prove: $\Diamond\alpha \vdash_{S5D} \Box\Diamond\alpha$

1		$\Diamond\alpha$	Assumption
2	□	$\Diamond\alpha$	1 SR-□(5)
3		$\Box\Diamond\alpha$	2 □ I

For systems with '□' as primitive and '◊' as derived, we can amend the strict reiteration rule using Duality. We begin with $\sim\Box\alpha$ instead of α and strictly "reiterate" $\sim\Box\alpha$

Strict Reiteration for '∼□' (5)

	$\sim\Box\alpha$	Already derived
□	$\vdots$	
	$\sim\Box\alpha$	SR-∼□ (5)

This is easily seen to be a derived rule given the original system with SR (5).

Strict Reiteration for '∼□' (5) as a derived rule

	$\sim\Box\alpha$	Already derived
	$\Diamond\sim\alpha$	Duality
□	.	
	.	
	.	
	$\Diamond\sim\alpha$	SR (5)
	$\sim\Box\alpha$	Duality

We can show that the SR(B) and SR(4) rules are derived rules given SR(5) and Strong ◊ Introduction (which reflect the euclidean and reflexive nature of accessibility, respectively). For SR(B), this is easy to see. When we are given a sentence α on a line, we can use Strong ◊ Introduction to get $\Diamond\alpha$ and then apply the SR(5) rule.

Strict Reiteration (B) as a derived rule

1	α	Already derived
2	$\Diamond\alpha$	Strong $\Diamond$ I
3	□ | $\vdots$	
4	| $\Diamond\alpha$	SR(5)

The proof that SR(4) is a derived rule is much more complicated. This is mirrored by the greater complexity of the derivation of transitivity of a relation from its being reflexive and euclidean. In particular, two different uses of Π Introduction had to be made in the former derivation.

Strict Reiteration for '□' (4) as a derived rule

$\Box\alpha$	Already derived
$\Diamond\Box\alpha$	Strong $\Diamond$ I
□ | $\vdots$	
| $\Diamond\Box\alpha$	SR-□(5)
| | $\sim\Box\alpha$	Assumption
| | $\Diamond\sim\alpha$	Duality
| | □ | $\Diamond\sim\alpha$	SR-□(5)
| | | $\sim\Box\alpha$	Duality
| | $\sim\Diamond\Box\alpha$	$\sim\Diamond$ I
| | $\Diamond\Box\alpha$	Reiteration
| $\Box\alpha$	$\sim$ E
| □ | α	SR-□

Given that the semantical system for *S5I* can be generated by making accessibility an equivalence relation, it should be the case that SR-□(5) is a derived rule given □ E, SR(B), and SR-□(4). Respectively, they reflect reflexivity, symmetry, and transitivity. Indeed, it can be shown that SR-□(5) is a derived rule given just the two Strict Reiteration rules. This is most easily demonstrated using Duality and the derived *SD* rule Double Negation.

Strict Reiteration for '□' (5) as a derived rule

	$\Diamond\alpha$	Already derived
□	$\vdots$	
□ |	$\sim\Diamond\alpha$	Assumption
□ |	$\Box\sim\alpha$	Duality
□ |	$\Diamond\Diamond\alpha$	SR(B)
□ |	$\Diamond\Diamond\sim\sim\alpha$	Double Negation
□ |	$\Diamond\sim\Box\sim\alpha$	Duality
□ |	$\sim\Box\Box\sim\alpha$	Duality
□ | □ | □	$\sim\alpha$	SR-□(4)
□ | □	$\Box\sim\alpha$	□ I
□ |	$\Box\Box\sim\alpha$	□ I
□	$\Diamond\alpha$	~ E

Given the simplification introduced by the no-accessibility semantical system *S5I*, it might be wondered whether a simpler derivational system could be based on it. The simplified semantics has the property that the truth-value of a modal sentence is invariant across worlds.[7] If $\Diamond\alpha$ is true at a world **w**, then it is true at all worlds in **W**, since all that it required is that α be true at some world in **W**. Similar reasoning holds for $\Box\alpha$ and $\alpha \dashv\!\!\!\!\!\!\;3\, \beta$.

To reflect this situation, we can still use restricted scope lines to represent other worlds, but now the only restriction is that a reiterated sentence must be modal. To generate a simplified derivational system *S5D*, we can consolidate the all the Strict Reiteration rules into a single rule: If α is a modal sentence, it may be strictly reiterated across any number of restricted scope lines. We can call this rule SR(M), or Strict Reiteration for Modalities.

Strict Reiteration for Modalities

	α	Already derived
□	$\vdots$	
□	α	SR(M)

Provided: α is a sentence whose main operator is modal

This blanket rule gives us the effect of the basic Strict Reiteration rule for '□' (which is also the rule for *KD*), since $\Box\alpha$ can be strictly reiterated by SR(M) and □ Elimination applied to yield α, which is what **SR**-□ requires.

[7]Recall that a modal sentence is one whose main operator is modal.

Strict Reiteration for '$\Box$' as a derived rule

$\vert\ \Box\alpha$	Already derived
$\vert\ \Box\ \vert\ \vdots$	
$\vert\quad\ \vert\ \Box\alpha$	SR(M)
$\vert\quad\ \vert\ \alpha$	$\Box$ E

The proof of Strict Reiteration for '$\Box$'(4) as a derived rule is straightforward. There are two uses of SR(M) followed by a use of $\Box$ Elimination.

Strict Reiteration for '$\Box$'(4) as a derived rule

$\vert\ \Box\alpha$	Already derived
$\vert\ \Box\ \vert\ \vdots$	
$\vert\quad\ \vert\ \Box\alpha$	SR(M)
$\vert\quad\ \vert\ \Box\ \vert\ \vdots$	
$\vert\quad\ \vert\quad\ \vert\ \Box\alpha$	SR(M)
$\vert\quad\ \vert\quad\ \vert\ \alpha$	$\Box$ E

For SR(B), if α occurs in a step, Weak $\Diamond$ Introduction may be applied, and the resulting $\Diamond\alpha$ may be strictly reiterated by SR(M).

Strict Reiteration (B) as a derived rule

$\vert\ \alpha$	Already derived
$\vert\ \Diamond\alpha$	Strong$\Diamond$ I
$\vert\ \Box\ \vert\ \vdots$	
$\vert\quad\ \vert\ \Diamond\alpha$	SR(M)

This shows that SR(M) yields all the less-powerful Strict Reiteration rules as derived rules. It remains to be shown that the original Strict Reiteration rules yield SR(M). But we have already shown Fitch's version of Strict Reiteration for '$\Box$' (4) allows the strict reiteration of $\Box\alpha$ intact across a strict scope line. And the Strict Reiteration for '$\Box$' (5) similarly allows the strict reiteration of $\Diamond\alpha$ intact.

Exercise. Show why sentences whose main operator is '⥽' may be reiterated intact across strict scope lines, given the rules for *KD* through *S5D*.

3 The Axiom System *S5*

The axiom system *S5* is obtained by adding to the axiom schemata of *T* the further axiom schema:

$\vdash_{S5} \Diamond\alpha \supset \Box\Diamond\alpha.$

This axiom is valid in *S5I*, by the same reasoning that was used to validate the corresponding semantical entailment.

4 Applications of the *S5*-Systems

The *S5*-systems are extremely strong—too much so for most applications. They seem only to represent logical modalities.

4.1 Alethic Modal Logic

The original application of the *S5*-systems was made by Carnap, who understood the '□' to represent logical necessity. His way of understanding the meaning of logical necessity is close to our notion of what holds by virtue of "laws of logic."

> The concept of logical necessity . . . seems to be commonly understood in such a way that it applies to a proposition *p* if and only if the truth of *p* is based on purely logical reasons and not dependent upon the contingency of facts; in other words, if the assumption of not-*p* would lead to a logical contradiction, independent of facts. (*Meaning and Necessity*, p. 174)

Carnap defended his choice of *S5I* as the semantical system for logical necessity and possibility by claiming that the fact of L-truth is itself based on purely logical reasons. Here is a quotation from *Meaning and Necessity*. Carnap uses different notation than ours. It should be kept in mind that his "S_2" is our *S5I*, and his 'N' is our '□.'

> Let 'A' be an abbreviation for an L-true sentence in S_2 (for example, 'Hs ∨ ~Hs'). Then 'N(A)' is true, according to [semantical rule] 39-1. And, moreover, it is L-true, because its truth is established by the semantical rules which determine the truth and thereby the L-truth of 'A', together with the semantical rule for 'N', say 39-1. Thus, generally, if 'N(. . .)' is true, then 'NN(. . .)' is true; hence any sentence of the form 'N*p* ⊃ NN*p*' is true. This constitutes an affirmative answer to the controversial question mentioned in the beginning. (*Meaning and Necessity*, Second edition, p. 174)

Another argument in favor of *S5I* might be found in the fact that it has a semantics with no accessibility relation. It could be held that accessibility itself represents a "contingency of facts" that helps determine truth-values. It might seem that from the standpoint of pure logic, there is nothing which dictates the superiority of one non-universal kind of accessibility over any other (or of there being no restrictions at all). In *S5I*, accessibility can be dispensed with altogether.

In a semantical system based on essentially on accessibility, there is a split between the possibility represented by the truth-values of sentences at the possible worlds (which really represent possible interpretations) and the possibility represented by the *accessiblilty* of possible worlds. If logical possibility is to be the most general kind of possibility (which seems plausible), then the more limited possibility expressed by the '◊' in systems weaker than the *S5*-systems is something different from logical possibility. Two worlds might have the same truth-values for all its sentences, in which case there is no logical difference between them, and one might be "possible" and the other not relative to a given world.

Reasoning in the other direction shows the unsuitability of the *S5*-systems for representing hypothetical necessity. Given the availability of a formulation of the semantics without accessibility, there is no way for the modal operators to express a condition that might or might not hold. Thus, for example, we can make no distinction in *S5* between logical, metaphysical, and physical possibility. The necessity operator expresses an absolute, not a hypothetical, notion of necessity in the *S5*-systems. The truths of necessity are truths of logic and are not contingent in any way.

4.2 Conditional Logic

With respect to strict implication, the most important result in the *S5*-systems is that $\{\Diamond(\alpha \mathrel{-\!\!3} \beta)\} \vDash_{S5I} \alpha \mathrel{-\!\!3} \beta$. In terms of representing logical implication, there seems to be no good reason to reject this entailment, so long as one is tolerant of the "paradoxes." (This is the result we reached in considering the *S4*-systems as a representation of strict implication.) If we accept the argumentation of Carnap, the possibility of a strict implication holding is a possibility that presumably would hold by virtue of the logical form of $\alpha \mathrel{-\!\!3} \beta$. So it seems that if it is possible that α strictly implies β, then α really does strictly imply β.

If $\alpha \mathrel{-\!\!3} \beta$ is supposed to express a relation that holds at worlds that meet a certain condition, then we run up against the objection that the *S5*-systems are not suitable for representing hypothetical necessity and possibilities.

4.3 Deontic Logic

It is clear that the *S5*-systems are far too strong for a deontic interpretation. They contain the objectionable elements of the *T*-, *B*-, and *S4*-systems, as well as a further consequence, $\{\mathrm{PO}\alpha\}\vDash_{S5I}\mathrm{O}\alpha$, that is problematic for some conceptions of obligation and permission. Let us suppose that it is legally permissible in the United States in 2005 that there be an obligation to perform some military service. (It may be permissible because conscription is permitted by the Constitution.) That surely does not mean that it is in fact legally obligatory to perform some military service. On the other hand, in a system of absolute obligation, as might be found in Kant's "categorical imperative," the distinction between the obligatory and the permissibily obligatory might collapse.[8]

4.4 Doxastic Logic

If the *S5*-systems are understood as representing a logic of belief, they carry with them the result that $\{\mathrm{P}_{s,t}\mathrm{B}_{s,t}\alpha\} \vDash_{S5I}\mathrm{B}_{s,t}\alpha$ (as well as the undesirable results from the *T*- and *B*-systems). If it is compatible with what I believe that I believe that α, then I believe that α. This consequence looks dubious because of its equivalent, $\{\sim\mathrm{B}_{s,t}\alpha\} \vDash_{S5I}\mathrm{B}_{s,t} \sim\mathrm{B}_{s,t}\alpha$. Is it the case that a logical saint believes of everything that he does not believe, that he does not believe it?'

Ordinary people do not satisfy this condition, for much of what we do not believe is the result of our not having conceived of various possible truths. Now the logical saint was introduced in order to accommodate the fact that given the semantical system *KI*, all valid sentences of *KI* are believed. (And similarly for the stronger systems.) So the saint would have to have a conception of all the sentences in the language, since a valid sentence could be any of them. Still, sainthood extends only as far as logical matters. Whether the saint believes anything whose truth is not dictated by the semantics does not seem to lie in the domain of logic.

This concern might still be overcome by a further consideration. Suppose one could survey all one's beliefs. Then what one does not believe is simply what is not found as the result of the survey. We granted in the last section that it is plausible that if the logical saint believes something, then he believes that he believes it. The reason was that it is plausible that a logical saint has complete command of what his beliefs are. Given what was just said, it would then be plausible that he has complete command of what his beliefs are not. Given the semantical system *S4I*, if he does not believe he believes something, then he does not believe it. And if he does not believe it, then it seems that we should say that he would recognize this and thus believe that he does not believe it.

[8]For the original statement of the categorical imperative, see Kant's *Groundwork of the Metaphysics of Morals*.

4.5 Epistemic Logic

Given that knowledge has a belief requirement, the result that might work in doxastic logic, $\{F_{s,t}K_{s,t}\alpha\} \vDash_{S5I} K_{s,t}\alpha$, might seem to work in epistemic logic as well. But now consider the equivalent $\{\sim K_{s,t}\alpha\} \vDash_{S5I} K_{s,t} \sim K_{s,t}\alpha$. The reason one may fail to know something might be that he has a false belief. But we cannot presume that the logical saint is able to detect all of his false beliefs and hence know what he does not know as the result of having them. He would have to be omniscient—a epistemological god rather than a mere logical saint.

4.6 Temporal Logic

The *S5*-systems are also too strong as a logic of futurity. Aside from the problems with the *T*- and *B*-systems, they give the result that if at some time in the future α will always be the case, α now will always be the case. When I am 60 years old, it will always be the case that I will be older than 59 years old, but since I am still in my 50s, it will not always be the case (starting from the present) that I am older than 59 years old. Similar remarks hold for the past. More generally, the reduction of the modalities into only two does not seem appropriate for time as we commonly conceive it. Each time is a unique standpoint with respect to its own future and past.