Module 12
Other Useful Normal Modal Systems

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Contents

At this point, we have found that some of the informal interpretations of modalities are suitably represented by systems that have been covered. In particular, system *D* is plausible as a formalization of the logic of obligation and system *S5* seems right for a logic of logical possibility and necessity. It also is somewhat plausible for representing logical implication. Hypothetical modalities, we have noted, may be represented by any of the systems, depending on the nature of the conditions governing the modality. In this module, we will examine some other normal systems that some have found suitable as logics of belief, knowledge, and temporality.

1 Logics of Belief: *DI*, *KD4I*, *KD45I*

In previous modules, it has been shown why the consequence-relations characteristic of the *K*-, *D*-, *S4*- and *S5*-systems may be plausible as representations of inferences involving the beliefs of idealized believers or "logical saints". The *K*-systems make belief closed under logical consequence. The *D*-systems require that what is believed be compatible with the body of one's beliefs.

The only two consequence-relations that are not plausible for the idealized notion of belief we are working with are those characteristic of *T* and *B*. These are related in the sense that they both involve non-modal sentences, which we can understand informally as representing what actually is the case. The beliefs of our "logical saint" cannot be expected to conform to "matters of fact" in general. All that is expected is that the saint have full command of logic and of what his beliefs are and are not.

The *D*-systems that have already been discussed may be considered to be the best way to capture the '$B_{s,t}$' operator as representing what ought to be believed. One ought to believe all logical truths, one ought to believe what follows from what one believes, and if one ought to believe something, then it is permissible for one to believe it.

The jump to the characteristic *S4*-entailment can be considered from the standpoint of an obligation to believe or from that of an idealized believer. We might want to say that if one ought to believe (as a matter of logic) that α, then one ought to believe (as a matter of logic) that one ought to believe that α.

For the idealized believer, transitivity requires that he be ideally introspective as well as logically omniscient. A system built on transitivity assumes that the idealized believer has an inventory of all that he believes, and moreover believes that he believes just those things. One might want to stop at this point and take the best doxastic systems to be the *KD4I*-systems, where accessibility is serial and transitive.

An argument was made in the the previous module that if the logical saint has total command of what is in his belief system, then he should be able to tell which of all possible beliefs his belief system lacks. Someone who is moved by this argument might wish to endorse the *KD5*-systems, where accessibility is serial, transitive, and euclidean.

Since each of these three systems has been formed on the basis of conditions on the accessibility-relations that have already been exaimined, there is nothing further to note about them here.

2 A Logic of Knowledge: *S4.2I*

Our investigation of the representation of knowledge showed that the *S4*-systems are most adequate among the normal systems covered in the previous modules. A number of systems stronger than the *S4*-systems but weaker than the *S5*-systems have been proposed as logics of knowledge. Here, we will take a look at one such semantical system, *S4.2I*. To obtain this system, we add to the restrictions of reflexivity and transitivity the requirement that accessibility by *convergent*:[1]

R is **convergent** iff
$(\Pi \mathbf{x})(\Pi \mathbf{y})(\Pi \mathbf{z})((\mathbf{Rxy} \wedge \mathbf{Rxz}) \rightarrow (\Sigma \mathbf{w})(\mathbf{Ryw} \& \mathbf{Rzw}))$.

If the accessibility relation branches out to two worlds, there is a world accessible to each of the two at which the relation converges. The characteristic entailment relation for *S4.2* is $\{\Diamond\Box\alpha\} \vDash_{S\mathbf{4.2I}} \Box\Diamond\alpha$.

The derivational system *S4.2D* is produced by allowing the Strict Reiteration, intact, of $\Diamond\Box\alpha$ across one restricted scope line.

Exercise. Show how the derivational system yields the characteristic derivation-relation for that system.

A meta-logical derivation shows that convergence yields the characteristic *S4.2I* entailment relation.

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[1]The semantics for S4.2 is due to Allen Hazen.

Sketch of a semantical proof that: $\{\Diamond\Box\alpha\} \vDash_{S4.2I} \Box\Diamond\alpha$

1	$v_I(\Diamond\Box\alpha, w) = T$	Assumption
2	$Rww_1 \wedge Rww_2 \rightarrow (\Sigma w_i)(Rw_1w_i \wedge Rw_2w_i)$	R is convergent
3	$(\Sigma w_i)(Rww_i \wedge v_I(\Box\alpha, w_i) = T)$	1 **SR**-$\Diamond$C
4	$\quad Rww_1 \wedge v_I(\Box\alpha, w_1) = T$	Assumption
5	$\quad\quad Rww_2$	Assumption
6	$\quad\quad Rww_1$	3 $\wedge$ E
7	$\quad\quad Rww_1 \wedge Rww_2$	5 6 $\wedge$ I
8	$\quad\quad (\Sigma w_1)(Rw_2w_i \wedge Rw_2w_i)$	2 7 $\rightarrow$ E
9	$\quad\quad\quad Rw_1w_3 \wedge Rw_2w_3$	Assumption
10	$\quad\quad\quad v_I(\Box\alpha, w_1) = T)$	4 $\wedge$ E
11	$\quad\quad\quad Rw_1w_3 \rightarrow v_I(\alpha, w_3) = T$	10 **SR**-$\Box$C
12	$\quad\quad\quad Rw_1w_3$	9 $\wedge$ E
13	$\quad\quad\quad v_I(\alpha, w_3) = T$	11-12 $\rightarrow$ E
14	$\quad\quad\quad Rw_2w_3$	9 $\wedge$ E
15	$\quad\quad\quad Rw_2w_3 \wedge v_I(\alpha, w_3) = T$	13 14 $\wedge$ I
16	$\quad\quad\quad (Rw_2w_3 \wedge v_I(\alpha, w_3) = T) \rightarrow v_I(\Diamond\alpha, w_2) = T$	15 **SR**-$\Diamond$
17	$\quad\quad\quad v_I(\Diamond\alpha, w_2) = T$	15 16 $\rightarrow$ E
18	$\quad\quad v_I(\Diamond\alpha, w_2) = T$	8 9-17 Σ E
19	$\quad Rww_2 \rightarrow v_I(\Diamond\alpha, w_2) = T$	5-18 $\rightarrow$ I
20	$\quad v_I(\Box\Diamond\alpha, w) = T$	19 **SR**-$\Box$
21	$v_I(\Box\Diamond\alpha, w) = T$	4 5-20 Σ E

Wolfgang Lenzen has proposed that *S4.2I* is a plausible system for representing knowledge.[2] In *S4.2I*, we have $\{F_{s,t}K_{s,t}\alpha\} \vDash_{S4.2I} K_{s,t}F_{s,t}\alpha$. From the fact that it is epistemically possible that one knows that α, it follows that one knows that it is epistemically possible that α. It might be useful to compare this with the result from *S5I* that has aleady been rejected as too strong: $\{F_{s,t}K_{s,t}\alpha\} \vDash_{S5I} K_{s,t}\alpha$. The mere epistemic possibility of one's knowing that α surely does not require that one knows that α. The *S4.2* inference is weaker. All that is known "automatically" from the epistemic possibility of one's knowing that α is one's knowing that α is epistemically possible.

If a case can be made for this inference, it must be based on the "logical sainthood" of the knower. So far, this notion has been pushed to require knowledge of logical consequences of what one knows (system *KI*) and to require knowledge of the fact that one knows what one knows (system *S4I*). That is, in each case, we have a consequence of the logical saint's knowing something. But the present case is not of this kind, as

[2]"*Epistemologische Betrachtungen zu [S4,S5]*", *Erkenntnis* 14 (1979), 33-56.

can be seen from an equivalent formulation of the characteristic *S4.2I* entailment: $\{\sim K\sim K\alpha\} \vDash_{S4.2I} KF\alpha$.[3] It might well be asked whether ignorance of one's knowing something generates knowledge. We have already rejected the similar entailment in *S5I*: $\{\sim K\sim K\alpha\} \vDash_{S5I} K\alpha$.

The answer depends on the specific fact of which our logical saint is ignorant. He may be ignorant of the fact that he is ignorant of α because he believes that α is true, when it is not, and hence he does not know that α. In the case of the *S4.2I* consequence, however, even if α is false, α may still be compatible with what he knows. So this sort of counterexample does not make $KF\alpha$ false. But we still need a reason to think it is true when $FK\alpha$ is true.

Such a reason may be forthcoming given the "traditional" definition of knowledge as true belief that is justified. If we use 'J' as a "justification" operator, we can formalize this definition of knowledge.

"Traditional" Definition of Knowledge
$K\alpha =_{Df} \alpha \ \& \ B\alpha \ \& \ J\alpha$.

Now suppose that for all the logical saint knows, he knows that α: $FK\alpha$. Then for all he knows, α is true, he believes α, and he is justified in believing that α: $F(\alpha \ \& \ B\alpha \ \& \ J\alpha)$. Since the 'F' operator distributes over conjunction (in the *K*-systems and their extensions), we can say that for all the logical saint knows, α; for all he knows he believes that α, and for all he knows he is justified in believing that α: $F\alpha \ \& \ FB\alpha \ \& \ FJ\alpha$.

We want to motivate the conclusion that as a result, $KF\alpha$, which by the definition is equivalent to $F\alpha \ \& \ BF\alpha \ \& \ JF\alpha$. Now the supposition of $FK\alpha$ and the definition of knowledge has already yielded $F\alpha$. It is plausible to say that if for all the logical saint knows, he believes that α, $FB\alpha$, then he believes that α is true for all he knows, $BF\alpha$. This is plausible, that is, on the assumption that the logical saint is introspectively omniscient, so that he knows exactly what he believes. In that case, if $FB\alpha$, then in fact the logical saint really believes α. He would be rational enough also to believe that α is compatible with what he knows, $BF\alpha$.[4]

Finally, there is some reason to think that one has similar command over what one is justified in believing. This is so long as we take justification to be "subjective" or an "internal matter" that is accessible to the logical saint. Then he should know what he is justified in believing, and if for all he knows he is justified in believing α, then he really is justified in believing that α. Again, we can conclude that to be rational, he would also have to be justified in believing that for all he knows, α. So given the assumptions that have been made, $F(\alpha \ \& \ B\alpha \ \& \ J\alpha)$ implies $F\alpha \ \& \ BF\alpha \ \& \ JF\alpha$. The needed assumptions are very powerful ones, and so the motivation for accepting the *S4.2* consequence-relation is very much theory-dependent.

Lenzen uses a quite different approach to motivate the use of the system. He makes an unusual proposal to define belief in terms of knowledge.

Lenzen's Definition of Belief
$B\alpha =_{Df} \sim K\sim K\alpha$

This definition allows the presence of both knowledge- and belief-operators in the syntax of the modal language, but this time goverened by all the rules for the formal language. From the definition, Lenzen draws some interesting results.

First, it can be shown, using reflexivity, that $\{K\alpha\} \vDash_{S4.2I} B\alpha$. This is easiest to see from the fact that from $K_{s,t}\alpha$ one can derive $\sim K\sim K\alpha$ using Strong $\Diamond$ Introduction to get $FK\alpha$ and then using Duality to get the belief operator. The next result is that $\{B\alpha\} \vDash_{S4.2} KB\alpha$, which is what we relied on in the earier argument motivating the use of *S4.2I* to represent knowledge. Finally, we have $\{B_{s,t}\alpha\} \vDash_{4.2} P_{s,t}K_{s,t}\alpha$.

Exercise. Use a semantical argument or a derivation (along with Lenzen's definition of belief) to show that

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[3]For ease of comprehension, we shall drop the subscripts for the rest of this section.

[4]If one accepts this argument, then one should strengthen the corresponding doxastic logic to *KD4.21* by adding convergence to seriality and transitivity.

these two consequence-relations hold.

These results are plausible with respect to the beliefs of a logical saint, and this provides some motivation for the use of *S4.2* to represent knowledge. But the motivation is only as strong as the plausibility of the definition of belief. It does seem pretty clear that when a logical saint rationally believes something α, then he does not know that he is ignorant of it. He does not know this because in fact he believes he does in fact know that α; otherwise, what does his belief that α amount to? The correctness of the other half of the definition is not self-evident. Lenzen understands belief in terms of ignorance of one's ignorance. Since belief involves some kind of positive commitment, the definition in this direction initially seems misguided.

To attack this issue, we should ask why one would be ignorant of his own ignorance that α. To answer this question, we might ask how one can be ignorant in the first place. If we allow with the "traditional" definition that knowledge is justified true belief, there are three ways in which one can be ignorant. First, one may be ignorant because α is not true. Now if α is not true, the subject either believes that it is true, or not. If he believes that α is true, then no counter-example can be constructed from the case. If he does not believe that α is true, then he does not know that α is true. But in this case, the falsehood of α is irrelevant. Given what we have said about the transparency of belief for the logical saint, he would not be ignorant of his ignorance on this basis; on the contrary, he would be aware of his ignorance. This takes care of the second reason for ignorance as well. The third reason is that the subject is not justified in his belief. But again, we can argue that the logical saint would not be ignorant of his lack of justification, and hence would not be ignorant of his ignorance of α.

So how could a subject be ignorant of his own ignorance? Again, there are three ways in which a person can fail to know that he is ignorant. It may be that he is not really ignorant in the first place, that is, it is $K\alpha$ actually holds. But on the traditional definition, this means that the logical saint believes that α. It may be that the subject does not believe that he is ignorant. For a logical saint, this lack of belief would be the result of rational deliberation: he does not believe he is ignorant because be believes that α is true. (The other alternative is that he has no commitment to whether he is ignorant or not, and we can assume that the logical saint would not find himself in this situation.) Finally, it may be that the logical saint is not justified in believing that he is ignorant. Again, this is either through inattention to the matter or as a result of rational deliberation. Assuming that the logical saint has rationally deliberated, it is reasonable to conclude that he does not think he is justified in thinking that he is ignorant because he actually believes that α. So in every case where the saint is ignorant of his ignorance that α, he believes that α.

It must be pointed out once again that these arguments rely heavily on assumptions about what a logical saint would be like. *S4.2I* is not the system of choice for theories of knowledge which repudiate these assumptions. And there are many that do.

3   Logics of the Future and the Past: *K4.3I*, *S4.3I*, and *S4.3.1*

We have seen that the possible worlds accessible to a given world can be understood as representing future times, and the worlds to which a given world is accessible can be understood to represent past times. We can begin to represent a logic of the future and the past with the system *KI*, since the property of always being the case in the future is closed under the relation of logical consequence. Because the seriality of the time-sequence is controversial, we will only require accessibility to be transitive, as we found transitivity to be a desirable feature for representing the relation between a time and past as well as future times. The resulting system will be called *K4I*.

There are other properties of futurity which are not captured by *K4I*, however. One such property is that time is *weakly connected*: for any two moments in the future, either they are the identical, or one is in the

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future of the other.[5]

    R is **weakly connected** iff
$$(\Pi x)(\Pi y)(\Pi z)((\mathbf{Rxy} \wedge \mathbf{Rxz}) \to (\mathbf{y} = \mathbf{z} \vee (\mathbf{Ryz} \vee \mathbf{Rzy}))).$$

The system *K4.3WI* is the result of adding this condition on accessibility to *K4I*. The characteristic entailment relation of *K4.3I*, due to E. J. Lemmon, is:

$$\{\sim \Box((\alpha \wedge \Box\alpha) \supset \beta)\} \vDash_{K4.3I} \Box((\beta \wedge \Box\beta \supset \alpha).$$

The axiom system *K4.3W* is formed by adding to the axiom system *K* the axiom schema for *S4* as well as the following axiom schema:

$$\vdash_{K4.3W} \Box((\alpha \wedge \Box\alpha) \supset \beta) \vee \Box((\beta \wedge \Box\beta) \supset \alpha).$$

This is not a schema that lends itself to an intuitive reading. Perhaps an equivalent formulation will be slightly more perspicuous:

$$\vdash_{K4.3W} \Box((\beta \vee \sim\alpha) \vee \sim\Box\alpha) \vee \Box((\alpha \vee \sim\beta) \vee \sim\Box\beta)).$$

We can put the matter in terms of a temporal interpretation. If, at some future time, α and $G\alpha$ are true, and β is false, then at each future time, either α is true or β is false at that time, or at some time later than that time, α is true.

    We may reason informally to this conclusion as follows (using times in place of worlds, with the temporal interpretation of the modal operators in mind). Suppose that at some time $\mathbf{t}_1$ in the future of $\mathbf{t}$, α and $\sim\beta$ are both true. Now let $\mathbf{t}_2$ be an arbitrary time in the future of $\mathbf{t}$. By the weak connectiveness property, either $\mathbf{t}_1$ is the same as $\mathbf{t}_2$, or $\mathbf{t}_1$ is after $\mathbf{t}_2$, or $\mathbf{t}_2$ is after $\mathbf{t}_1$.

    If $\mathbf{t}_1 = \mathbf{t}_2$, then it is the case that α is true at $\mathbf{t}_2$ (and also that $\sim\beta$ is true at $\mathbf{t}_2$). Then $(\alpha \vee \sim\beta) \vee F\sim\beta$ is true at $\mathbf{t}_2$. If $\mathbf{t}_2$ is in the future of $\mathbf{t}_1$, then since $G\alpha$ is true at $\mathbf{t}_1$, α is true at some future time $\mathbf{t}_2$. So $(\alpha \vee \sim\beta) \vee F\sim\beta$ is true at $\mathbf{t}_2$. Finally, if $\mathbf{t}_2$ is earlier than $\mathbf{t}_1$, then since $\sim\beta$ is true at $\mathbf{t}_1$, $F\sim\beta$ is true at $\mathbf{t}_2$, so that $(\alpha \vee \sim\beta) \vee F\sim\beta$ is true at $\mathbf{t}_2$. So in all three cases, $(\alpha \vee \sim\beta) \vee F\sim\beta$ is true at $\mathbf{t}_2$. Since the choice of $\mathbf{t}_2$ is arbitrary, it follows that at $\mathbf{t}$, $G((\alpha \vee \sim\beta) \vee F\sim\beta)$ is true.

    We can see that the characteristic entailment holds (in the disjunctive form) from the following, somewhat abbreviated, meta-logical derivation. (In some cases, the converses of semantical rules for Sentential Logic operators have been used. 'WC' indicates the condition of **R** being weakly connected.)

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[5]In some of the literature, this kind of relation is called "piece-wise connected."

Sketch of a semantical proof that: $\{\sim \Box((\beta \lor \sim\alpha) \lor \sim\Box\alpha)\} \vDash_{K4.3WI} \Box((\alpha \lor \sim\beta) \lor \sim\Box\beta))$

1	$\mathbf{v_I}(\sim\Box((\beta \lor \sim\alpha) \lor \sim\Box\alpha), \mathbf{w}) = \mathbf{T}$	Assumption
2	$\mathbf{v_I}(\Box((\beta \lor \sim\alpha) \lor \sim\Box\alpha), \mathbf{w}) = \mathbf{F}$	1 **SR**-$\sim$C
3	$(\Sigma\mathbf{w}_i)(\mathbf{Rww}_i \bigwedge \mathbf{v_I}((\beta \lor \sim\alpha) \lor \sim\Box\alpha, \mathbf{w}_i) = \mathbf{F})$	2 **SR**-$\Box$
4	$\mathbf{Rww}_1 \bigwedge \mathbf{v_I}(((\beta \lor \sim\alpha) \lor \sim\Box\alpha), \mathbf{w}_1) = \mathbf{F}$	Assumption
5	$\mathbf{v_I}(\beta, \mathbf{w}_1) = \mathbf{F} \bigwedge \mathbf{v_I}(\sim\alpha, \mathbf{w}_1)) = \mathbf{F} \bigwedge \mathbf{v_I}(\sim\Box\alpha), \mathbf{w}_1) = \mathbf{F}$	4 **SR**-$\lor$C
6	$\mathbf{Rww}_2$	Assumption
7	$\mathbf{w}_1 = \mathbf{w}_2 \bigvee \mathbf{Rw}_1\mathbf{w}_2 \bigvee \mathbf{Rw}_2\mathbf{w}_1$	3 4 WC
8	$\mathbf{w}_1 = \mathbf{w}_2$	Assumption
9	$\mathbf{v_I}(\sim\alpha, \mathbf{w}_1) = \mathbf{F}$	5 $\bigwedge$ E
10	$\mathbf{v_I}(\alpha, \mathbf{w}_1) = \mathbf{T}$	9 **SR**-$\sim$C
11	$\mathbf{v_I}(((\alpha \lor \sim\beta) \lor \sim\Box\beta), \mathbf{w}_1) = \mathbf{T}$	10 **SR**-$\lor$
12	$\mathbf{Rw}_1\mathbf{w}_2$	Assumption
13	$\mathbf{v_I}(\sim\Box\alpha), \mathbf{w}_1) = \mathbf{F}$	5 $\bigwedge$ E
14	$\mathbf{v_I}(\Box\alpha), \mathbf{w}_1) = \mathbf{T}$	13 **SR**-$\sim$C
15	$\mathbf{Rw}_1\mathbf{w}_2 \bigwedge \mathbf{v_I}(\Box\alpha), \mathbf{w}_1) = \mathbf{T}$	12 14 $\bigwedge$ I
16	$\mathbf{v_I}(\alpha), \mathbf{w}_2) = \mathbf{T}$	15 **SR**-$\Box$C
17	$\mathbf{v_I}(((\alpha \lor \sim\beta) \lor \sim\Box\beta), \mathbf{w}_1) = \mathbf{T}$	16 **SR**-$\lor$
18	$\mathbf{Rw}_2\mathbf{w}_1$	Assumption
19	$\mathbf{v_I}(\beta, \mathbf{w}_1) = \mathbf{F}$	5 $\bigwedge$ E
20	$\mathbf{v_I}(\sim\beta, \mathbf{w}_1) = \mathbf{T}$	19 **SR**-$\sim$
21	$\mathbf{v_I}(((\alpha \lor \sim\beta) \lor \sim\Box\beta), \mathbf{w}_1) = \mathbf{T}$	21 **SR**-$\lor$
22	$\mathbf{v_I}(((\alpha \lor \sim\beta) \lor \sim\Box\beta), \mathbf{w}_1) = \mathbf{T}$	7-21 $\land$ E
23	$\mathbf{Rww}_2 \to \mathbf{v_I}(((\alpha \lor \sim\beta) \lor \sim\Box\beta), \mathbf{w}_1) = \mathbf{T}$	6-22 $\to$ I
24	$\mathbf{v_I}(\Box((\alpha \lor \sim\beta) \lor \sim\Box\beta), \mathbf{w}) = \mathbf{T}$	23 **SR**-$\Box$
25	$\mathbf{v_I}(\Box((\alpha \lor \sim\beta) \lor \sim\Box\beta), \mathbf{w}) = \mathbf{T}$	3 4-23 Σ E

The semantical system *K4.3WI* allows for frames in which two times are accessible to each other. This seems inappropriate for our common notion of time, as was remarked in the module on the **B**-systems. We would like accessibility to be an *anti-symmetrical* relation:

R is **anti-symmetrical** iff
$(\mathbf{Rw}_i\mathbf{w}_j \bigwedge \mathbf{Rw}_j\mathbf{w}_i) \to \mathbf{w}_i = \mathbf{w}_j$.

Krister Segerberg has shown that there is a method for constructing frames with the desired accessibility

relation, which are equivalent to the *KS4.3I* frames described here.[6] We will not here describe why this is the case, but only note that because of this result, *K4.3* emerges as an adequate system for representing linear time.

It was noted earlier that the *T*-systems seem a reasonable starting-point for a logic for operators that symbolize what is and always will be the case , and what is or sometime will be the case. (And corresponding relations for the past.) Transitivity would apply to such modalities as well, so we can move up to the *S4*-systems from the *T*-systems. We could add the *K4.3* axiom to get a corresponding system, but because of the reflexivity of accessibility, we can get the same effect from the following condition:

R is **connected** iff
$(\Pi \mathbf{x})(\Pi \mathbf{y})(\Pi \mathbf{z})((\mathbf{Rxy} \& \mathbf{Rxz}) \rightarrow (\mathbf{Ryz} \bigvee \mathbf{Rzy}))$.

The fact that connectedness follows from weak connectedness and reflexivity is easily shown.

Sketch of a proof that: $(\Pi \mathbf{x})\mathbf{Rxx} \rightarrow (\mathbf{y} = \mathbf{z}) \rightarrow (\mathbf{Ryz} \bigvee \mathbf{Rzy})$

1	$(\Pi \mathbf{x})\mathbf{Rxx}$	Assumption
2	$\mathbf{y} = \mathbf{z}$	Assumption
3	$\mathbf{Ryy}$	1 Π E
4	$\mathbf{Ryz}$	2 3 = E
5	$\mathbf{Ryz} \bigvee \mathbf{Rzy}$	4 $\bigvee$ I
6	$\rightarrow \mathbf{Ryz} \bigvee \mathbf{Rzy}$	4 $\bigvee$ I
7	$(\Pi \mathbf{x})\mathbf{Rxx}$	1-5 $\rightarrow$ I

The addition of the connectedness condition to the semantical system *S4I* yields the semantical system *S4.3I*, whose axiom schema is:

$\vdash_{S4.3} \Box(\Box\alpha \supset \beta) \vee \Box(\Box\beta \supset \alpha)$.

This axiom is easily seen to be equivalent to the following theorem:

$\vdash_{S4.3} \Diamond(\Box\alpha \wedge \sim\beta) \supset \Box(\alpha \vee \Diamond\sim\beta)$.

To this theorem corresponds the consequence relation:

$\{\Diamond(\Box\alpha \wedge \sim\beta)\} \vdash_{S4.3} \Box(\alpha \vee \Diamond\sim\beta)$..

The semantical version of this consequence relation can be shown to hold in the semantical system *S4.3I*.

Exercise. Assuming that the accessibility relation is reflexive, transitive, and connected, sketch a semantical proof that $\{\Diamond(\Box\alpha \wedge \sim\beta)\} \vDash_{S4.3} \Box(\alpha \vee \Diamond\sim\beta)$.

The *S4.3*-systems turn out to be able to represent only continuous time. To represent discrete time, a stronger system is needed. This system is known as *S4.3.1*, whose axiom schema is the following:

$\vdash_{S4.3.1} \Box(\Box(\alpha \supset \Box\alpha) \supset \alpha) \supset (\Diamond\Box\alpha \supset \alpha)$.

We will not discuss this system any further here.[7] Nor will we discuss how to represent time's coming to an end or not.[8]

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[6]For an excellent presentation of Segerberg's method, see Hughes and Cresswell, *A Companion To Modal Logic*, Chapter 5.

[7]For details, see Hughes and Cresswell, *A New Introduction to Modal Logic*, pp. 179-180. The system was originally called 'D' by Prior.

[8]See Hughes and Cresswell, *A New Introduction to Modal Logic*, Chapter 7.