Module 16
The *Q1R* Systems

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Contents

The weakest family of systems of Modal Predicate Logic is are based on the respective systems of Free Predicate Logic, adding to them rules to govern the behavior of modal operators. Where, '*x*' stands for the modal system in question, the systems treated in this module will be called *Q1RI-x* systems.[1]

1 The Semantical *Q1RI-x* Systems

In this section, we will spell out a framework for semantical systems that is compatible with treating the accessibility relation in any way we wish.

1.1 Frames and Interpretations

The key to adapting the Free Predicate Logic semantics to the modal semantics is the relativization of inner and outer domains to possible worlds. That is, each of the possible worlds will have a domain of existing objects, which counts as its inner domain. All other objects, which may be in the domains of other possible worlds, lie in that world's outer domain. This is done formally in a very simple way. A frame will contain a single domain **D** and a function **q**: $\mathbf{Fr}=\langle \mathbf{W}, \mathbf{R}, \mathbf{D}, \mathbf{q}\rangle$. The function **q** assigns to world **w** members of **D** as its domain $\mathbf{D^w}$. That is, $\mathbf{q(w)} = \mathbf{D^w}$. Since $\mathbf{q(w)} \subseteq \mathbf{D}$, we can say $\mathbf{D^w} \subseteq \mathbf{D}$. The inner domain of Free Predicate Logic for world **w** is then identified with $\mathbf{D^w}$.

The outer domain at a world is simply what remains in **D** (the set specified in the frame) when the objects in the inner domain are removed. In non-modal Free Predicate Logic, we indicated the outer domain as $\mathbf{D}'$. Now we can extend this notation to apply to the outer domain relative to the domain at a world, $\mathbf{D^{w}}'$. Then the outer domain at a world is just the result of removing all the objects in the domain at the world from the frame's domain: $\mathbf{D^{w}}' = \mathbf{D} - \mathbf{D^w}$. So for example we might have an interpretation on which D={1,2}, $\mathbf{D^{w_1}}$={1}, and D^{w_2}={2}. Then $D^{w_1}{}'$={2} and $D^{w_2}{}'$={1}.

Finally, an interpretation **I** will be a frame plus a valuation function: $\mathbf{I}=\langle \mathbf{W}, \mathbf{R}, \mathbf{D}, \mathbf{q}, \mathbf{v}\rangle$. The only difference between this kind of interpretation and the kind described in the last module is the presence of the **q** function which assigns domains to worlds.

1.2 Terms

We shall require that the value for a term (constant or parameter) at a world, $\mathbf{v_I(a,w)}$, be a member of the domain specified by the frame (or the "frame's domain"). This value is defined even if the designated object is in the outer domain of a world because in free logic constants and parameters are not assumed to designate objects that exist. In the modal system, the term might not designate an object existing at a world, i.e., an object in that world's inner domain, but it must designate at least what could be called a "possible object." The designation of termss by the valuation function is "rigid," in the sense that the function assigns the same member of the frame's domain at all worlds.

1.3 Predicates

As in the last module, the extension of an *n*-place predicate at a world is drawn from the frame's domain **D**: $\mathbf{v_I}(\mathbf{P}^n, \mathbf{w}) \subseteq \mathbf{D}^n$. An object may be in the extension of a predicate at a world even though it does not exist at that world, so that (semi-formally) ⟨Pegasus⟩ is in the extension of 'winged horse.' A term which designates an object not in $\mathbf{D^w}$ nonetheless designates an object in $\mathbf{D^{w}}'$.

[1] James Garson uses the name 'Q1R' for the axiomatic version of the systems to be developed here. It is related to another system, *Q1*, which will be treated in the next module.

A special semantical rule was needed in Free Predicate Logic for the predicate constant 'E.' This is easily adapted to modal semantics by requiring that the value of 'E' at world **w** be just the set of one-tuples whose members are in the domain of **w**, which is identified with the inner domain. That set is the same as the 1st Cartesian product of the inner domain, so we have: $\mathbf{v_I}(\mathrm{E},\mathbf{w}) \in \mathbf{D}^{\mathbf{w}1}$.

1.4 Identity

We will say that an identity sentence is true at a world **w** if and only if the valuation function assigns the same member of the frame's domain **D** to both terms flanking the identity sign. An identity sentence is false at **w** just in case the valuation function does not assign the same member of **D** to both terms. Due to Rigid Designation, the terms flanking the identity sign designate the same object at all worlds in the frame of an interpretation. Therefore, identity sentences are, if true on an interpretation, true at all worlds in the interpretations's frame. If false on an interpretation, they are false at all worlds in the frame of the interpretation. This fact preserves bivalence and truth-functionality for identity sentences.

1.5 Operators

Sentencess whose main connectives are modal or non-modal sentential operators are also treated exactly as in the semantical systems for non-modal Sentential logic.

1.6 Quantifiers

Quantified sentences are treated just as they are in Free Predicate Logic, except that sentences are evaluated at worlds. We want to say that a universally quantified sentence $(\forall \mathbf{x})\alpha(\mathbf{x}/\mathbf{u})$ is true at a world **w** just in case $\alpha(\mathbf{v}/\mathbf{u})$ true true on all variants of the valuation function which assign to **v** objects in the domain of **w**. Similarly, an existentially quantified sentence $(\forall \mathbf{x})\alpha(\mathbf{x}/\mathbf{u})$ is true at a world just in case the $\alpha(\mathbf{v}/\mathbf{u})$ is true for some variant of the valuation function which assigns to **v** an object in the domain of the world.

Semantical Rules for *Q1RI*

1. $\mathbf{v_I}(\mathbf{a},\mathbf{w}) \in \mathbf{D}$

2. $\mathbf{v_I}(\mathbf{u},\mathbf{w}) \in \mathbf{D}$

3. $(\Pi \mathbf{w}_i)(\Pi \mathbf{w}_j)((\mathbf{w}_i \in \mathbf{W} \wedge \mathbf{w}_j \in \mathbf{W}) \rightarrow \mathbf{v_I}(\mathbf{t}, \mathbf{w}_i) = \mathbf{v_I}(\mathbf{t}, \mathbf{w}_j))$

4. $\mathbf{v_I}(\mathbf{P}^n,\mathbf{w}) \subseteq \mathbf{D}^n$

5. $\mathbf{v_I}(\mathbf{P}^n\mathbf{t}_1 \ldots \mathbf{t}_n,\mathbf{w})=\mathbf{T}$ if and only if $\langle \mathbf{v_I}(\mathbf{t}_1,\mathbf{w}),\ldots,\mathbf{v_I}\mathbf{t}_1,\mathbf{w})\rangle \in \mathbf{v_I}(\mathbf{P}^n,\mathbf{w})$;
   $\mathbf{v_I}(\mathbf{P}^n\mathbf{t}_1 \ldots \mathbf{t}_n, \mathbf{w})=\mathbf{F}$ if and only if $\neg(\langle \mathbf{v_I}(\mathbf{t}_1,\mathbf{w}),\ldots,\mathbf{v_I}(\mathrm{t}_n,\mathbf{w})\rangle \in \mathbf{v_I}(\mathbf{P}^n,\mathbf{w}))$

6. $\mathbf{v_I}(\mathrm{E},\mathbf{w}) = \mathbf{D}^{\mathbf{w}1}$

7. $\mathbf{v}_I(\mathbf{t}_i=\mathbf{t}_j,\mathbf{w})=\mathbf{T}$ if and only if $\mathbf{v}_I(\mathbf{t}_i,\mathbf{w})=\mathbf{v}_I(\mathbf{t}_j,\mathbf{w})$;
   $\mathbf{v}_I(\mathbf{t}_i=\mathbf{t}_j,\mathbf{w})=\mathbf{F}$ if and only if $\neg(\mathbf{v}_I(\mathbf{t}_i,\mathbf{w})=\mathbf{v}_I(\mathbf{t}_j,\mathbf{w}))$

8. $\mathbf{v_I}((\forall\mathbf{x})(\alpha(\mathbf{x}/\mathbf{u}), \mathbf{w})=\mathbf{T}$ (where $\mathbf{u}$ is free for $\mathbf{x}$ in $\alpha(\mathbf{x})$) if and only if for all $\mathbf{d} \in \mathbf{D^w}$, $\mathbf{v}_\mathrm{I}[\mathbf{d}/\mathbf{u}](\alpha(\mathbf{u}), \mathbf{w}) = \mathbf{T}$;
   $\mathbf{v_I}((\forall\mathbf{x})(\alpha(\mathbf{x}/\mathbf{u}), \mathbf{w})=\mathbf{F}$ if and only if for some $\mathbf{d} \in \mathbf{D^w}$, $\mathbf{v}_\mathrm{I}[\mathbf{d}/\mathbf{u}](\alpha(\mathbf{u}), \mathbf{w}) = \mathbf{F}$.

9. $\mathbf{v_I}((\exists\mathbf{x})(\alpha(\mathbf{x}/\mathbf{u}), \mathbf{w})=\mathbf{T}$ (where $\mathbf{u}$ is free for $\mathbf{x}$ in $\alpha(\mathbf{x})$) if and only if for some $\mathbf{d} \in \mathbf{D^w}$, $\mathbf{v}_\mathrm{I}[\mathbf{d}/\mathbf{u}](\alpha(\mathbf{u}), \mathbf{w}) = \mathbf{T}$;
   $\mathbf{v_I}((\exists\mathbf{x})(\alpha(\mathbf{x}/\mathbf{u}))=\mathbf{F}$ if and only if for all $\mathbf{d} \in \mathbf{D^w}$, $\mathbf{v}_\mathrm{I}[\mathbf{d}/\mathbf{u}](\alpha(\mathbf{u}), \mathbf{w}) = \mathbf{F}$.

Summary of Systems *Q1RI-x*
Frames as defined for *Q1RI-x*
v defined for □, ◊, and non-modal operators as for the basic system
v defined for ∀ and ∃ based on $\mathbf{D}^w$ as the inner domain
Modal restrictions on **R** for system *x*
Rigid Designation for constants and parameters

1.7 Semantical Properties and Relations

We assert, without proof, that the properties of *Modal Quantificational Bivalence* and *Modal Quantificational Truth-Functionality* hold for the *Q1RI-x* systems. The semantical rules are set up in such a way that each sentence gets one and only one truth-value. The relation of *Modal Quantificational Semantical Entailment* is the same as with the semantical systems for Modal Sentential Logic. For all worlds on interpretations, if on an arbitrary interpretation each member of a set of sentences is true is true at a world, then the entailed sentence is true at that world. *Modal Quantificational Validity*, the limiting case of Modal Semantical Entailment is defined as before as well. Finally, there are no changes to make in the definition of *Modal Quantificational Semantical Consistency*.

Because the *Q1RI-x* systems are based on Free Predicate Logic, there are some non-entailments that do not involve modality. For example, $\{(\forall\mathbf{x})\alpha(\mathbf{x})\} \nvDash_{Q1RI-x} \alpha(\mathbf{a}/\mathbf{x})$ and $\{\alpha(\mathbf{a}/\mathbf{x})\} \nvDash_{Q1RI-x} (\exists\mathbf{x})\alpha(\mathbf{x})$.

1.8 Quantifiers and Modalities in the *Q1RI-x* Systems

As noted in the previous module, much of the interest in Modal Predicate Logic involves the representation of modalities *de re* and *de dicto*. Now that we have semantical machinery in place, we can see how the two kinds of modality are related in the *Q1RI-x* systems.

The schemata $(\forall\mathbf{x})\Box\alpha(\mathbf{x}/\mathbf{u}) \supset \Box(\forall\mathbf{x})\alpha(\mathbf{x}/\mathbf{u})$ and $\Diamond(\exists\mathbf{x})\alpha(\mathbf{x}/\mathbf{u}) \supset (\exists\mathbf{x})\Diamond\alpha(\mathbf{x}/\mathbf{u})$ are known in the literature as variations of the *Barcan formula*, as the latter was an axiom schema of the first published axiomatization of Modal Predicate Logic.[2] Any sentence that is an instance of one schema is an instance of the other. The schemata $\Box(\forall\mathbf{x})\alpha(\mathbf{x}/\mathbf{u}) \supset (\forall\mathbf{x})\Box\alpha(\mathbf{x}/\mathbf{u})$ and $(\exists\mathbf{x})\Diamond\alpha(\mathbf{x}/\mathbf{u}) \supset \Diamond(\exists\mathbf{x})\alpha(\mathbf{x}/\mathbf{u})$ are the forms of the *converse Barcan formula*. Again, any instance of one is an instance of the other.

Exercise. Using Duality and Quantifier Negation (from the derivational system for *PLI*), prove the claims just made that any instance of one schema is equivalent to an instance of the other.

Given our emphasis on the consequence-relation in modal logic, we will be looking at what will be called the *Barcan Consequences* and the *Converse Barcan Consequences*: :

[2]Ruth Barcan, "A Functional Calculus of the First Order Based on Strict Implication", *Journal of Symbolic Logic* 11 (1946) 1-16. The existential sentence was used in the original paper, as the '◊' was generally used as a primitive operator at that time. Barcan's system was based on Lewis's *S2*.

Barcan Consequences
$\{(\forall \mathbf{x})\Box\alpha(\mathbf{x}/\mathbf{u})\} \vDash \Box(\forall \mathbf{x})\alpha(\mathbf{x}/\mathbf{u})$,
$\{\Diamond(\exists \mathbf{x})\alpha(\mathbf{x}/\mathbf{u})\} \vDash (\exists \mathbf{x})\Diamond\alpha(\mathbf{x}/\mathbf{u})$,

Converse Barcan Consequences
$\{\Box(\forall \mathbf{x})\alpha(\mathbf{x}/\mathbf{u})\} \vDash (\forall \mathbf{x})\Box\alpha(\mathbf{x}/\mathbf{u})$,
$\{(\exists \mathbf{x})\Diamond\alpha(\mathbf{x}/\mathbf{u})\} \vDash \Diamond(\exists \mathbf{x})\alpha(\mathbf{x}/\mathbf{u})$.

Informally, the Barcan Consequence with the universal quantifier and the necessity operator states that if everything in the domain of a world is such that if 'u' designates it, then necessarily '$\alpha(\mathbf{u})$' is true, then necessarily, for each member of the domain of the world, if 'u' designates it, then '$\alpha(\mathbf{u})$' is true. So for example, if everything is such that necessarily it is either round or not round, then necessarily, everything is round or not-round.

The failure of the Barcan Consequence in *Q1RI-S5* established by a counter-example, using an interpretation I. Let **R** be a universal relation and (in present terms) $D^{w_1}=\{1\}$ and $D^{w_2}=\{1,2\}$. Finally, suppose that $\langle 1\rangle \in v_I(F,w_1)$, $\langle 1\rangle \in v_I(F,w_2)$, and $\langle 2\rangle \notin v_I(F,w_2)$. On this interpretation, $v_I[1/u](Fu,w_2) = \mathbf{T}$, and w_2 is the only world accessible to w_1, $v_I[1/u](\Box Fu,w_1) = \mathbf{T}$. And since 1 is the only member of the domain of w_1, we have it that $v_I((\forall x)\Box Fx,w_1) = \mathbf{T}$. On the other hand, since $v_I[2/u](Fu,w_2) = \mathbf{F}$, in which case, $v_I((\forall x)Fx,w_2) = \mathbf{F}$. In that case, $v_I(\Box(\forall x)Fx,w_1) = \mathbf{F}$.

We can visualize this reasoning by beefing up the modal truth-tables used in the Modal Sentential Logic modules. We will indicate the domain at the world and the extension of the key predicates at that world. For quantified sentences, we give a value for parameters in the context in which sentences containing them are evaluated.

w_1	$\leftrightarrows$	w_2
D: $\{1\}$		D: $\{1,2\}$
$F:\{\langle 1\rangle\}$		$F:\{\langle 1\rangle\}$
$(\forall x)\Box Fx$		
T		
$\Box Fu$ [1/u]		
T		
Fu [1/u]		Fu [1/u]
T		**T**
		Fu [2/u]
		F
		$(\forall x)Fx$
		F
$\Box(\forall x)Fx$		
F		

To show the failure of the converse Barcan Consequence in *Q1RI-S5*, we first reverse the domains. That is, $D^{w_1}=\{1,2\}$ and $D^{w_2}=\{1\}$. Then we stipluate once more that the extension of 'F' includes only $\langle 1\rangle$ at both worlds, in which case it does not contain $\langle 2\rangle$ at w_1. Then we get the following table.

$\curvearrowleft$		$\curvearrowleft$
w_1	$\leftrightarrows$	w_2
D: {1,2}		D: {1}
$F : \langle 1 \rangle$		$F : \langle 1 \rangle$
$\Box(\forall x)Fx$		
T		
$(\forall x)Fx$		$(\forall x)Fx$
T		**T**
Fu [1/u]		Fu [1/u]
T		**T**
Fu [2/u]		Fu [2/u]
T		**F**
$\Box Fu$ [2/u]		
F		
$(\forall x)\Box Fx$		
F		

The same frame can be used to show the invalidity of '$(\exists x)\Box x = a$,' given that $v_I(a, w_2)$=2. Then $v_I[1/u]u = a$,w_2)=**F**. Hence, $v_I[1/u](\Box u = a$,w_1)=**F**. Since v[1/u] is the only variant of v that is relevant at w_1, $v_I((\exists x)\Box x = a$,$\mathbf{w}_1$)=**F**.

$\curvearrowleft$		$\curvearrowleft$
w_1	$\leftrightarrows$	w_2
D: {1}		D: {1,2}
$a : 2$		$a : 2$
		$u = a$ [1/u]
		F
$\Box u = a$[1/u]		
F		
$(\exists x)\Box x = a$		
F		

Exercise. Show that '$\Box(\exists x)x = a$' is invalid in any *Q1RI-x* system.

Because of rigid designation, there are some semantical entailments of modal identity sentences. Sentences of the form $\mathbf{a}_i$=$\mathbf{a}_j$ semantically entail sentences of the form $\Box\mathbf{a}_i$=$\mathbf{a}_j$ are valid. The reason for this is that the value of any constant is taken from **D** and so the value of a constant at a world is not affected by its having a more limited domain. This affects only the valuation of sentences with quantifiers. The sentence 'a=a' is true at all worlds for this reason as well.

Exercise. Show the validity in all *Q1R-x* systems of '(∀x)(∀y)(x=y ⊃ □x=y).' [Hint: this requires the use of variants of variants.]

Another result following from Rigid Designation involves the existential quantifier. Although sentences of the form $(\exists\mathbf{x})\alpha(\mathbf{x}/\mathbf{u})$ is not valid in the system, sentences of the form $(\exists\mathbf{x})\Box\alpha(\mathbf{x}/\mathbf{u})$ are entailed by them. This result for systems based on *KI* is illustrated in the following table. Here we will introduce some new notation, prefixing a meta-logical quantifier before the mention of a member of the domain. Thus, we will write ‘$\Sigma\mathbf{d}^1$’ to indicate some member of the domain of $\mathbf{w}^1$ and ‘$\Pi\mathbf{d}^1$’ to indicate all the members of the domain of $\mathbf{w}^1$. Thus, the second line reads that ‘$\mathbf{u} = \mathbf{a}$’ is true for at least one member of $\mathbf{d}^1$ when ‘$\mathbf{u}$’ designates it. The key move is from the second to the third row. Because of Rigid Designation, all identity sentences have the same truth-value at all worlds.

w_1	$\xrightarrow{*}$	w_2
$(\exists\mathbf{x})(\mathbf{x} = \mathbf{a})$ **T**		
$\mathbf{u} = \mathbf{a}\ [\Sigma\mathbf{d}^1/\mathbf{u}]$ **T**		
		$\mathbf{u} = \mathbf{a}\ [\Sigma\mathbf{d}^1/\mathbf{u}]$ **T**
$\Box\mathbf{u} = \mathbf{a}\ [\Sigma\mathbf{d}^1/\mathbf{u}]$ **T**		
$(\exists\mathbf{x})\Box\mathbf{x} = \mathbf{a}$ **T**		

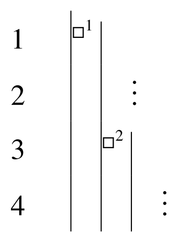
2 The Derivational Systems *Q1RD-x*

The derivational systems *Q1RD-x* are obtained by using the derivational rules of system *x* (e.g., system *T*) along with a modalized version of derivational rules for Free Predicate Logic. (Here we will call the rules ‘∀ Introduction (Q1R), etc.)

2.1 Indexed Restricted Scope Lines

The rules specific to *Q1RD-x* introduce a new kind of restricted scope line, one which combines the restrictions of the scope lines for modal operators with those for quantifiers. In the derivational rules for Modal Sentential Logic, we have restricted scope lines which limit the use of reiteration. In the rules for Predicate Logic, there are “barrier” lines which prevent the reiteration of any sentence containing the parameter that flags the scope line. We want to be able to reflect in our use of parameters the domain of the world where the parameter first appears.

The first step in the combination is to put an index number on the box to the left of a modal restricted scope line. The first such scope line gets the number 1, the next scope line within its scope gets the number 2, etc. So we could have the following setup.



The next step is to require that any parameter flanking a restricted scope line for quantifiers be given the index of the highest-numbered modal operator in whose scope it lies. Thus we can have:

$$
\begin{array}{ll}
1 & \Box^1 \mid \\
2 & \quad \mid \quad \vdots \\
3 & \quad \mid\ \Box^2 \mid \\
4 & \quad \mid \quad \mid \quad \vdots \\
5 & \quad \mid \quad \mid\ u^2 \mid \\
6 & \quad \mid \quad \mid \quad \mid \quad \vdots
\end{array}
$$

Finally, we will restrict the application of quantifier rules to sentences containing parameters which are indexed to the highest numbered modal restricted scope line in whose scope the sentence lies. Before stating the rules formally, we will give a simple example of how they can be used.

To Prove: $\vdash_{Q1RD-S5} \Box(\forall x)x = x$

$$
\begin{array}{llll}
1 & \Box^1 \mid u^1 \mid & u^1 = u^1 & = \mathrm{I} \\
2 & \quad \mid & (\forall x)x = x & 1\ \forall\ \mathrm{I}\ (Q1RD) \\
3 & & \Box(\forall x)x = x & 1\text{-}2\ \Box\ \mathrm{I}
\end{array}
$$

This derivation reflects our semantical reasoning. Consider an arbitrary world **w** and an arbitrary world $\mathbf{w}_1$ accessible to **w**. At such a world, for any parameter **u**, '**u** = **u**' is true, because '**u**' always designates the same member of the domain. The parameter '**u**' picks out a member of the domain of $\mathbf{w}_1$, which is reflected in the tagging of 'u_1' in the derivation above. Thus at $\mathbf{w}_1$, '$(\forall x)x = x$' is true, and therefore, $\Box(\forall x)x = x$ is true at **w**.

Restricted Scope Line (*Q1R*)

$$
\begin{array}{l}
\Box^n \mid \\
\quad\ \mid \\
\quad\ \mid
\end{array}
$$

If the restricted scope line is not in the scope of any other restricted scope line, $n = 1$.
If the restricted scope line is in the immediate scope of a restricted scope line of index m, $n = m + 1$.

Barrier Scope Line (*Q1R*)

$$
\begin{array}{l}
\Box^n \mid \\
\quad\ \mid \quad \vdots \\
\quad\ \mid\ u^n \mid \\
\quad\ \mid \quad\ \mid \quad \vdots
\end{array}
$$

If the barrier scope line is not in the scope of any restricted scope line, n = 0.
If the barrier scope line is in the immediate scope of a restricted scope line of index m, n=m.

The rule of Barrier Removal applies in *Q1RD-x* for the same reason it applies in *FPD*: if the parameter does not occur in a sentence, then there is nothing relevant for the barrier to screen.

We are now in a position to state revised versions of the rules for quantifiers where instantiation and generalization are made using parameters only.

2.2 Quantifier Rules

The rules for Universal Elimination are modeled on the rules of *FPD*. In fact, the rule for constants is the same, since parameters, and hence indices on parameters, are not involved in it. For instantiation to sentences with parameters, we will require that the parameter have the same index as the flag. The flag is supposed to restrict the parameters in a way that reflects the confinement of the values of parameters to the objects in the domain of a world. So if we instantiate a universal sentence, the parameter's index should be the same as the index on the '□' that represents the relevant world.

Universal Elimination for Parameters (*Q1R*)

$$
\begin{array}{l|l|ll}
\mathbf{v}^n & & (\forall \mathbf{x})\alpha(\mathbf{x}/\mathbf{u}^n) & \text{Already Derived} \\
 & & \vdots & \\
 & & \alpha(\mathbf{v}^n/\mathbf{u}^n) & \forall\ \mathrm{I}\ (Q1R)
\end{array}
$$

Provided that: No sentence containing **v** is reiterated across the barrier scope line.

Universal Elimination for Constants (*Q1R*)

$$
\begin{array}{|ll}
\vdots & \\
(\forall \mathbf{x})\alpha(\mathbf{x}/\mathbf{u}) & \text{Already Derived} \\
\vdots & \\
\mathrm{E}\mathbf{a} & \\
\vdots & \\
\alpha(\mathbf{a}/\mathbf{u}) & \forall\ \mathrm{I}\ (Q1R)
\end{array}
$$

The same considerations as just given motivate the rule of Universal Introduction. We should be able to generalize on parameters whose index reflects the domain of the world at which generalization is made, which in turn is represented by the index on the '□' flag.

Universal Introduction (*Q1R*)

$$
\begin{array}{l|lll}
 & \vdots & & \\
 & \mathbf{v}^n \mid & \vdots & \\
 & \phantom{\mathbf{v}^n} \mid & \alpha(\mathbf{v}^n/\mathbf{u}^n) & \\
 & (\forall \mathbf{x})\alpha(\mathbf{x}/\mathbf{u}^n) & & \forall\ \mathrm{I}\ (Q1R)
\end{array}
$$

Provided that: No sentence containing **u** is reiterated across the barrier scope line.

As with Universal Elimination, the version for constants Existential Introduction is the same as in *FPD*. The Introduction rule once again matches the index on the parameter with the index on the '□' flag.

Existential Introduction for Parameters (*Q1R*)

$$
\begin{array}{l|l|ll}
 & \mathbf{v}^n & \alpha(\mathbf{v}^n/\mathbf{u}^n) & \text{Already Derived} \\
 & & \vdots & \\
 & & (\exists \mathbf{x})\alpha(\mathbf{x}/\mathbf{u}^n) & \exists\ \mathrm{I}\ (Q1R)
\end{array}
$$

Provided that: No sentence containing **v** is reiterated across the barrier scope line.

Existential Introduction for Constants (*Q1R*)

$$
\begin{array}{|ll}
\alpha(\mathbf{a}/\mathbf{u}) & \text{Already Derived} \\
\vdots & \\
\mathrm{E}\mathbf{a} & \\
\vdots & \\
(\exists \mathbf{x})\alpha(\mathbf{x}/\mathbf{u}) & \exists\ \mathrm{I}\ (Q1R)
\end{array}
$$

Provided that: No sentence containing **v** is reiterated across the barrier scope line.

Finally, Existential Elimination is a straightforward adaptation of the *FPD* rule, with the index on the assumption being appropriately indexed.

Existential Elimination (*Q1R*)

$$
\begin{array}{|lll}
(\exists \mathbf{x})\alpha(\mathbf{x}/\mathbf{u}^n) & & \text{Already derived} \\
\mathbf{v}^n \mid & \alpha(\mathbf{v}^n/\mathbf{u}^n) & \text{Assumption} \\
\phantom{\mathbf{v}^n} \mid & \vdots & \\
\phantom{\mathbf{v}^n} \mid & \beta & \\
\beta & & \exists\ \mathrm{E}
\end{array}
$$

Provided that: no sentence containing **v** is reiterated across the restricted scope line,
v does not occur in β.

To illustrate the effects of these rules, we will display unsuccessful attempts at derivations of entailments that do not hold in the *Q1R-x* systems.

Attempt to prove: $\{\Box(\forall x)Fx\} \vdash_{Q1RD-K} (\forall x)\Box Fx$

1	$\mid\ \Box(\forall x)Fx$	Assumption
2	$\mid\ {}^{u^0}\mid\ \Box(\forall x)Fx$	1 Reiteration
3	$\mid\ \mid\ {}^{\Box^1}\mid\ (\forall x)Fx$	2 SR-$\Box$
4	$\mid\ \mid\ \mid\ {}^{u^1}\mid\ (\forall x)Fx$	3 Reiteration
5	$\mid\ \mid\ \mid\ \mid\ Fu^1$	4 $\forall$ E (*Q1R*)
6	$\mid\ \mid\ \mid\ Fu^1$	5 Misapplication of BR
7	$\mid\ \mid\ \Box Fu^1$	$\Box$ I
8	$\mid\ (\forall x)\Box Fx$	4 Misapplication of $\forall$ I (*Q1R*)

There are two problems with this attempted derivation. The first is that Barrier Removal cannot be used because the parameter in the flag occurs in the formula. The second problem is that the generalization at the last step requires the index 0, but the index is instead 1.

Attempt to prove: $\{(\forall x)\Box Fx\} \vdash_{Q1RD-K} \Box(\forall x)Fx$

1	$\mid\ (\forall x)\Box Fx$	Assumption
2	$\mid\ {}^{u^0}\mid\ (\forall x)\Box Fx$	1 Reiteration
3	$\mid\ \mid\ \Box Fu^0$	2 $\forall$ E (*QR1*)
4	$\mid\ \mid\ {}^{\Box^1}\mid\ Fu^0$	3 SR-$\Box$
5	$\mid\ \mid\ \mid\ {}^{u^1}\mid\ Fu^0$	4 Reiteration
6	$\mid\ \mid\ \mid\ (\forall x)Fx$	5 Misapplication of $\forall$ I (*Q1R1*)
7	$\mid\ \mid\ \Box(\forall x)Fx$	1 2-6 $\Box$ I
8	$\mid\ \Box(\forall x)Fx$	7 BR

The problem with this attempt at a derivation lies at step 4, where the index needed for generalization on the parameter '*u*' is one lower than it needs to be.

Attempt to prove: $\{(\exists x)\Box Fx\} \vdash_{Q1RD-K} \Box(\exists x)Fx$

1	$(\exists x)\Box Fx$	Assumption
2	$\quad \Box Fu^0$	Assumption
3	$\quad \Box^1 \quad Fu^0$	2 SR-$\Box$
4	$\quad\quad u^1 \quad Fu^0$	3 Misapplication of Reiteration
5	$\quad\quad\quad (\exists x)Fx$	4 Misapplication of $\exists$ I (*Q1R*)
6	$\quad\quad (\exists x)Fx$	5 BR
7	$\quad \Box(\exists x)Fx$	3-6 $\Box$ I
8	$\Box(\exists x)Fx$	1 2-7 $\exists$ E

The misapplication at step 4 is an illicit crossing of the barrier. The second is that the parameter '*u*' has the wrong index.

Attempt to prove: $\{(\exists x)\Diamond Fx\} \vdash_{Q1RD-K} \Diamond(\exists x)Fx$

1	$(\exists x)\Diamond Fx$	Assumption
2	$\quad \Diamond Fu^0$	Assumption
3	$\quad \Box^1 \quad Fu^0$	2 SR-$\Diamond$
4	$\quad\quad u^1 \quad Fu^0$	3 Misapplication of Reiteration
5	$\quad\quad\quad (\exists x)Fx$	4 Misapplication of $\exists$ I (*Q1R*)
6	$\quad\quad (\exists x)Fx$	5 BR
7	$\quad \Diamond(\exists x)Fx$	2 3-6 $\Diamond$ E
8	$\Diamond(\exists x)Fx$	1 2-7 $\exists$ E

The problems in this attempted derivation are the same: the parameter was reiterated across the barrier line and the index is not the correct one.

Attempt to prove: $\{\Diamond(\forall x)Fx\} \vdash_{Q1RD-K} (\forall x)\Diamond Fx$

1	$\Diamond(\forall x)Fx$	Assumption
2	$\quad \Box^1 \quad (\forall x)Fx$	1 SR-$\Diamond$
3	$\quad\quad u^1 \quad (\forall x)Fx$	2 Reiteration
4	$\quad\quad\quad Fu^1$	4 $\forall$ E (*Q1R*)
5	$\quad\quad Fu^1$	4 5 Misapplication of BR
6	$\quad \Diamond Fu^1$	1 2-6 $\Diamond$ E
7	$(\forall x)\Diamond Fx$	7 Misapplication of $\forall$ I

In this derivation, Barrier Removal is applied improperly at step 6, since the parameter in the flag occurs in the sentence brought out from beind the barrier. At the last step, the index on the parameter is incorrect.

As the *Q1RD-x* systems are rather weak, there are not many interesting derivations to examine. We can get some results with rules governing the existence predicate 'E.' In non-modal Free Predicate Logic, there are rules which allow derivations involving constants. We can carry those rules over directly. That is, we can allow Universal Instantiation and Existential Introduction within a restricted scope line without any indexing. Because 'E**a**' is a non-modal sentence, it cannot be reiterated across a restricted scope line, so existence is always asserted "locally" at a given world.

Here is one result we can get, which is an instance of Closure applied to Free Predicate Logic, in which $\{(\forall x)Fx, \mathrm{E}a\} \vdash_{FPL} Fa$.

To prove: $\{\Box(\forall x)Fx, \Box\mathrm{E}a\} \vdash_{Q1RD-K} \Box Fa$

1			$\Box(\forall x)Fx$	Assumption
2			$\Box\mathrm{E}a$	Assumption
3	$\Box^1$		$(\forall x)Fx$	1 SR (*Q1R*)
4			$\mathrm{E}a$	2 SR (*Q1R*)
5		u^1	$(\forall x)Fx$	3 Reiteration
6			$\mathrm{E}a$	4 Reiteration
7			Fa	5 6 $\forall$ E (*Q1R*)
8			Fa	7 BR
9			$\Box Fa$	1 2 5-8 $\Box$ I

Note that Universal Generalization cannot be applied to the last step, so we cannot derive '$(\forall x)\Box Fx$,' which would be an instance of the converse Barcan Consequence.

2.3 Modal Rules for Identity Sentence

We will introduce two rules for the derivational system which reflect the fact that semantically, constants and parameter function as rigid designators. This has the consequence that every identity sentence is true at all worlds or false at all worlds. In the latter case, the negation of the identity sentence is true at all worlds.

2.3.1 Strict Reiteration for Identity Sentences

We will include in the derivational system *Q1RD-x* a rule a of Strict Reiteration for '=,' which functions in the same way as Strict Reiteration for '$\Box$.'

Strict Reiteration for '='

		$\mathbf{t}_i = \mathbf{t}_j$	Already Derived
$\Box^n$		$\vdots$	
		$\mathbf{t}_i = \mathbf{t}_j$	SR-=

Here is a derivation that makes use of this rule.

To prove: $\{(\exists x)x = a\} \vdash_{Q1RD-K} (\exists x)\Box x = a$

1	$(\exists x)x = a$	Assumption
2	u^1 $\quad u^1 = a$	Assumption
3	$\Box^1$ $\quad\quad u^1 = a$	3 SR-=
4	$\quad \Box u^1 = a$	2 3 □ I
5	$\quad (\exists x)\Box x = a$	4 ∃ I (*Q1R*))
6	$(\exists x)\Box x = a$	1 2-5 ∃ E (*Q1R*))

2.3.2 Strict Reiteration for Negated Identity Sentences

The following rule reflects the fact that if two terms do not designate the same object at a given world, they do not do so at any possible world.

Strict Reiteration for '∼ ='

	$\sim\mathbf{t}_i = \mathbf{t}_j$	Already Derived
$\Box^n$	$\quad \vdots$	
	$\quad \sim\mathbf{t}_i = \mathbf{t}_j$	SR-∼ =

3 Systems Stronger than *Q1R-x*

The weakest family of systems of Modal Predicate Logic is the family of *Q1R-x* systems. They provide the most flexibility in dealing with non-existent objects. Their only requirement is that at least one thing must exist in the domain of each of the worlds. The price of this flexibility is that very few quantifier/modality interactions are required by the systems. But it should be noted that strength can be added by placing restrictions on the accessibility relation. The system *Q1R-x-S5* is a very strong system modally, but that modal strength does not force us into existential requirements.

Even greater flexibility might be sought in a system of Modal Predicate Logic. As we saw above, one might wish that constants not be rigid designators, for example. Such systems have been developed, but they are considerably more complicated than the systems considered here.[3] One might, perhaps for different reasons, which to make identity contingent, so that what is identical is not necessarily identical. This leads to yet other complications.[4]

In both cases, what is required is a change in the semantics. Constants and variables must designate "intensional objects" or "individual concepts" rather than the standard objects required in the semantical systems we have been examining. A good case can be made that this is needed to produce the most desirable systems of modal logic.[5] But an examination of these systems will not be made here.

[3] See Garson discussion of *Q3* in "Quantification in Modal Logic".
[4] See Hughes and Cresswell, *A New Introduction to Modal Logic*, pp. 334-335.
[5] See especially Garson on this issue.

In the next module, we will consider two families of systems stronger than the *Q1R-x* systems. One of them yields the Barcan Consequences and the other yields its Converse. In the final module, we will treat a system which yields both of these consequences. In such systems, most, but not all, modalities *de re* and *de dicto* are equivalent.