Philosophy 134 Exercise Solutions

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Modal Sentential Logic I

Exercise. Show how to define the circle in terms of the fish-hook. For a much more difficult challege, try to define either one-place operator in terms of strict implication.

$\alpha \prec \beta =_{Df} \sim \Diamond(\alpha \,\&\, \sim \beta)$
$\sim \Diamond(\alpha \,\&\, \sim \beta) =_{Df} \sim (\alpha \circ \sim \beta)$
Therefore, $\alpha \prec \beta =_{Df} \sim (\alpha \circ \sim \beta)$

$\sim \alpha \prec \alpha =_{Df} \sim \Diamond(\sim \alpha \,\&\, \sim \alpha)$
$\sim \Diamond(\sim \alpha \,\&\, \sim \alpha)$ is equivalent to $\sim \Diamond \sim \alpha$ [by Idempotence]
$\sim \Diamond \sim \alpha =_{Df} \Box\alpha$
Therefore, $\sim \alpha \prec \alpha$ is equivalent to $\Box\alpha$

$\sim(\alpha \prec\sim \alpha) =_{Df} \sim\sim \Diamond(\alpha \,\&\, \sim\sim \alpha)$
$\sim\sim \Diamond(\alpha \,\&\, \sim\sim \alpha)$ is equivalent to $\sim\sim \Diamond(\alpha \,\&\, \alpha)$[by Double Negation]
$\sim\sim \Diamond(\alpha \,\&\, \alpha)$ is equivalent to $\Diamond(\alpha \,\&\, \alpha)$ [by Double Negation]
$\Diamond(\alpha \,\&\, \alpha)$ is equivalent to $\Diamond\alpha$[by Idempotence]
Therefore, $\sim(\alpha \prec\sim \alpha)$ is equivalent to $\Diamond\alpha$

Exercise. Give an argument for why truth-functional entailment for *SL* carries over to entailment in all frames of *MSL*.

A set of sentences $\{\gamma_1, \cdots, \gamma_n\}$ truth-functionally entails a sentence α if and only if on all interpretations on which each γ_i is true, α is true. An interpretation in the semantics for *SL* is an assignment of truth-values to all the sentence-letters of *SL*. A set of sentences $\{\gamma_1, \cdots, \gamma_n\}$ entails α in a frame just in case for any **I** based on **Fr** and any **w** in **W** in **Fr**, if $\mathbf{v}_I(\gamma_1, \mathbf{w})$=**T** and, $\cdots$, and $\mathbf{v}_I(\gamma_n, \mathbf{w})$=**T**, then $\mathbf{v}_I(\alpha, \mathbf{w})$=**T**. Truth-values are assigned to sentence-letters at worlds in frames in exactly the same way as they are assigned to sentences letters in *SL*-interpretations. The truth-definitions for non-modal sentences are made in exactly the same way in the semantics for *SL* and in the semantics for *MSL*. Now suppose $\{\gamma_1, \cdots, \gamma_n\} \models_{SL} \alpha$ and $\mathbf{v}_I(\gamma_i, \mathbf{w})$=**T** for an arbitrary world **w** in an arbitrary frame, for all the members of $\{\gamma_1, \cdots, \gamma_n\}$. Since $\mathbf{v}_I(\gamma_i, \mathbf{w})$ is based on exactly the same determinations as the truth-value of γ_i in the semantics for *SL*, $\mathbf{v}_I(\alpha, \mathbf{w})$ will be given the same value as with *SL*, namely **T**. Since this holds for an aribitrary world and frame, $\{\gamma_1, \cdots, \gamma_n\} \vDash_{Fr} \alpha$.

Exercise: Give a definition of *consistency in a frame* which corresponds to the notion of truth-functional consistency. A set of sentences is truth-functionally consistent if and only if there is a row on a truth-table which makes all the sentences in that set true.

The definition from Sentential Logic does not carry over precisely, because in Modal Sentential Logic, truth is always relative to a world. So it seems reasonable to say that a set of sentences is consistent in a frame when all the sentences are assigned **T** at some world in the frame by some interpretation based on the frame. The fact that on an interpretation, all the sentences might be assigned **F** at a given world does not show that the set of sentences is inconsistent, any more than the fact that they are assigned **F** on a row of a truth-table shows that it is inconsistent.

Exercise. From '□P' and '□(P ⊃ Q)' we can derive '□Q'. This is equivalent to deriving '∼ ◊P' from '∼ ◊Q' and '∼ ◊(P & ∼Q)' Explain why these two derivations are equivalent.

By the definitional equivalences, '□P' is equivalent to '∼ ◊ ∼P', '□(P ⊃ Q)' is equivalent to '∼ ◊ ∼(P ⊃ Q)', and '□Q' is equivalent to '∼ ◊ ∼Q'. Within a restricted scope line, we can derive 'Q' from 'P' and 'P ⊃ Q'. By the derivational rules for *SD*, we can therefore derive '∼P' from '∼Q' and '(P ⊃ Q)' inside a strict scope line. 'P ⊃ Q' is derviationally equivalent to '∼(P & ∼Q)'. So the use of the Strict Reiteration rule for necessity yields an inference from 'P' and 'P ⊃ Q' to 'Q' which is equivalent to the way the use of the Strict Reiteration rule for Impossibility yields a derivation from '∼Q' and '∼(P & ∼Q)' to '∼P. And the use of the respective Introduction rules yields the desired results.

Exercise. Give an example of an entailment and derivation that hold for *K* and *KD* but not in *SL* and *SD*, respectively. Then show why closure holds in that case.

{□(P & Q)}$\vDash_K$□P, {□(P & Q)}$\vdash_K$□P. These do not hold in the semantical or derivational systems for *SL* because the component sentences contain modal operators that are not part of the syntax of *SL*.

Closure yields the following results: {□□(P & Q)} $\vDash_K$ □□P, {□□(P & Q)} $\vdash_K$ □□P.

For the entailment, suppose that for an arbitrary world **w** on an interpretation **I**, $\mathbf{v_I}$(□□(P & Q), **w**)=**T**. Then at all accessible worlds $\mathbf{w}_i$, $\mathbf{v_I}$(□(P & Q), $\mathbf{w}_i$)=**T**. And so at all worlds $\mathbf{w}_j$ accessible to $\mathbf{w}_i$, $\mathbf{v_I}$((P & Q), $\mathbf{w}_j$)=**T**. From the truth-definition of the conjunction symbol, we have it that $\mathbf{v_I}$(P, $\mathbf{w}_j$)=**T**. Because $\mathbf{w}_j$ is aribtrary, for all worlds accessible to $\mathbf{w}_i$, $\mathbf{v_I}$(□P, $\mathbf{w}_i$)=**T**. By the same reasoing, $\mathbf{v_I}$(□□P, **w**)=**T**. Since the choice of **w** and **I** are arbitrary, {□□(P & Q)}$\vDash_K$□□P, which was to be proved.

Here is the derivation.

1	$\Box\Box(P \,\&\, Q)$	Assumption
2	$\Box(P \,\&\, Q)$	1 Strict Reiteration
3	$P \,\&\, Q$	2 SR
4	P	
5	$\Box P$	3-4 $\Box$ Introduction
6	$\Box\Box P$	2-5 $\Box$ Introduction

Exercise. Prove the following semantic equivalence: $\Diamond\alpha$ is equivalent to $\sim\Box\sim\alpha$.

On an arbitrary interpretation **I** and arbitrary world **w** in **I**,

$\mathbf{v}_I(\Diamond\alpha, \mathbf{w})=\mathbf{T}$ iff
for some $\mathbf{w}_i$ such that $\mathbf{Rww}_i$, $\mathbf{v}_I(\alpha, \mathbf{w}_i)=\mathbf{T}$ iff
for some $\mathbf{w}_i$ such that $\mathbf{Rww}_i$, $\mathbf{v}_I(\sim\alpha, \mathbf{w}_i)=\mathbf{F}$ iff
for some $\mathbf{w}_i$ such that $\mathbf{Rww}_i$, it is not the case that $\mathbf{v}_I(\sim\alpha, \mathbf{w}_i)=\mathbf{T}$ iff
$\mathbf{v}_I(\Box\sim\alpha, \mathbf{w})=\mathbf{F}$ iff
$\mathbf{v}_I(\sim\Box\sim\alpha, \mathbf{w})=\mathbf{T}$. QED

Since the choice of interpretations and worlds is arbitrary, $\Diamond\alpha$ and $\sim\Box\sim\alpha$ have the same truth-value on all interpretations, and so they are semantically equivalent.

Exercise. Show that $\{\sim\Box\sim\alpha\}\vdash_K \Diamond\alpha$.

1	$\sim\Box\sim\alpha$	Assumption
2	$\sim\Diamond\alpha$	Assumption
3	$\sim\alpha$	2 Strict Reiteration ($\sim\Diamond$)
4	$\Box\sim\alpha$	3 $\Box$ Introduction
5	$\sim\Box\sim\alpha$	1 Reiteration
6	$\Diamond\alpha$	2-5 $\sim$ Elimination

Exercise. Justify the rule of Strict Reiteration ($\prec$) as a derived rule of *KD*.

This rule is only a derived rule if $\alpha\prec\beta$::$\Box(\alpha\supset\beta)$ is given as a replacement rule. Then the following derivation-schema holds.

$\alpha\prec\beta$	
$\Box(\alpha\supset\beta)$	Replacement Rule
.	
.	
.	
$\alpha\supset\beta$	Strict Reiteration

Exercise. Give a derivation-schema that shows closure of a single possibility-sentence over strict implication.

$\Diamond\alpha$	
$\alpha \prec \beta$	
$\quad\Vert\ \alpha$	Strict Assumption
$\quad\Vert\ \alpha \supset \beta$	SR($\prec$)
$\quad\Vert\ \beta$	$\supset$ Elimination
$\Diamond\beta$	$\Diamond$ Elimination

Exercise. Give a derviation to prove the following: $\{(\alpha \prec \beta)\&(\beta \prec \alpha)\} \vdash_K \Box(\alpha \equiv \beta)$.

$(\alpha \prec \beta)\&(\beta \prec \alpha)$	Assumption
$\alpha \prec \beta$	& Elimination
$\beta \prec \alpha$	& Elimination
$\quad\Vert\ \alpha \supset \beta$	Strict Reiteration ($\prec$)
$\quad\Vert\ \beta \supset \alpha$	Strict Reiteration ($\prec$)
$\quad\Vert\ \beta \equiv \alpha$	$\equiv$ Introduction
$\Box(\alpha \equiv \beta)$	$\Box$ I

Exercise. Symbolize the conditionals: "If p and q are consistent, then q and p are consistent", "If q and p are consistent, then it is possible that p be true", "If it is possible that p be true, then p is consistent with itself", as strict implications. Show that they hold in the semantics for K.

$(\alpha \circ \beta) \prec (\beta \circ \alpha)$. Consider an aribitrary world **w** in an arbitrary interpretation **I**. Suppose $\mathbf{v_I}(\alpha\circ\beta, \mathbf{w})$=**T**. Then at some accessible world $\mathbf{w}_i$, $\mathbf{v_I}(\alpha\&\beta, \mathbf{w}_i)$=**T**. So $\mathbf{v_I}(\alpha, \mathbf{w}_i)$=**T** and $\mathbf{v_I}(\beta, \mathbf{w}_i)$=**T**. Then $\mathbf{v_I}(\beta\&\alpha, \mathbf{w}_i)$=**T**. In that case, $\mathbf{v_I}(\beta\circ\alpha, \mathbf{w})$=**T**. Since the choice of world and interpretation is arbitrary, the entailment holds.

$(\alpha \circ \beta) \prec \Diamond\alpha$. Consider an aribitrary world **w** in an arbitrary interpretation **I**. Suppose $\mathbf{v_I}(\alpha \circ \beta, \mathbf{w})$=**T**. Then at some accessible world $\mathbf{w}_i$, $\mathbf{v_I}(\alpha\&\beta, \mathbf{w}_i)$=**T**. So $\mathbf{v_I}(\alpha, \mathbf{w}_i)$=**T**. Then $\mathbf{v_I}(\Diamond\alpha, \mathbf{w})$=**T**. Since the choice of world and interpretation is arbitrary, the entailment holds.

$\Diamond\alpha \prec (\alpha \circ \alpha)$. Consider an aribitrary world **w** in an arbitrary interpretation **I**. Suppose $\mathbf{v_I}(\Diamond\alpha, \mathbf{w})$=**T**. Then at some accessible world $\mathbf{w}_i$, $\mathbf{v_I}(\alpha, \mathbf{w}_i)$=**T**. So $\mathbf{v_I}(\alpha\&\alpha, \mathbf{w}_i)$=**T**. Then $\mathbf{v_I}(\alpha \circ \alpha, \mathbf{w})$=**T**. Since the choice of world and interpretation is arbitrary, the entailment holds.

Modal Sentential Logic II

Exercise. Give a justification for $\sim\Box$ Introduction using the semantics for D.

The occurrence of $\sim\alpha$ inside a restricted scope line indicates that it is taken to be true at an arbitary world $\mathbf{w}_i$ accessible to **w**, if there is one. Suppose $\mathbf{v}(\sim\alpha,\mathbf{w}_i)$=**T**. Then $\mathbf{v}(\alpha,\mathbf{w}_i)$=**F**. Now suppose further that $\mathbf{v}(\Box\alpha,\mathbf{w})$=**T**. Then if a world $\mathbf{w}_i$ is accessible to **w**, then, $\mathbf{v}(\alpha,\mathbf{w}_i)$=**T**. By the seriality of the accessibility relation, there is at least one world accessible to **w**. So at $\mathbf{w}_i$, $\mathbf{v}(\alpha,\mathbf{w}_i)$=**T** and $\mathbf{v}(\alpha,\mathbf{w}_i)$=$\mathbf{F}$, which is forbidden by the semantical rules. So the supposition that $\mathbf{v}(\Box\alpha,\mathbf{w})$=$\mathbf{T}$ is false, and so $\mathbf{v}(\Box\alpha,\mathbf{w})$=$\mathbf{F}$ and $\mathbf{v}(\sim\Box\alpha,\mathbf{w})$=$\mathbf{T}$. So we can end the restricted scope line and write down $\sim\Box\alpha$.

Exercise. Show that with Duality as a replacement rule and $\sim\Box$ Introduction, the rule of Weak $\Diamond$ Introduction can be replaced.

‖ .	
‖ .	
‖ .	
‖ α	
‖ $\sim\sim\alpha$	
$\sim\Box\sim\alpha$	$\sim\Box$ I
$\sim\sim\Diamond\alpha$	Duality
$\Diamond\alpha$	DN

Exercise. Show that Strong $\Diamond$ Introduction and $\Box$ Elimination are inter-derivable.

$\Box\alpha$	
| $\sim\alpha$	Assp
| $\Diamond\sim\alpha$	S $\Diamond$ I
| $\sim\Diamond\sim\alpha$	Duality
α	$\sim$ E

α	
| $\sim\Diamond\alpha$	Assp
| $\Box\sim\alpha$	Duality
| $\sim\alpha$ || α	R
$\Diamond\alpha$	$\sim$ E

Exercise. Prove the following *S4*-equivalences: $\Diamond\Diamond\alpha$ and $\Diamond\alpha$, $\Diamond\Box\Diamond\Box\alpha$ and $\Diamond\Box\alpha$, $\Box\Diamond\Box\Diamond\alpha$ and $\Box\Diamond\alpha$.

Since $\mathbf{R}$ is reflexive in the *S4* semantics, if $\Diamond\alpha$ is true at a world, then $\Diamond\Diamond\alpha$ is also true at that world, as it is itself an accessible world where $\Diamond\alpha$ is true. Now suppose that $\Diamond\Diamond\alpha$ is true at world $\mathbf{w}$. Then there is an accessible world $\mathbf{w}_i$ at which $\Diamond\alpha$ is true. In that case, there is a world $\mathbf{w}_j$ accessible to $\mathbf{w}_i$ at which α is true. Since accessibility is transitive in *S4*, $\mathbf{Rww}_j$, in which case $\Diamond\alpha$ is true at $\mathbf{w}$.

Suppose $\mathbf{v}(\Diamond\Box\alpha,\mathbf{w})$=$\mathbf{T}$. Then there is an accessible world $\mathbf{w}_i$ such that $\mathbf{v}(\Box\alpha,\mathbf{w}_i)$=$\mathbf{T}$. Suppose $\mathbf{Rw}_i\mathbf{w}_j$ and $\mathbf{Rw}_j\mathbf{w}_k$. By transitivity, $\mathbf{Rw}_i\mathbf{w}_k$. So $\mathbf{v}(\alpha,\mathbf{w}_k)$=$\mathbf{T}$. In that case, $\mathbf{v}(\Box\alpha,\mathbf{w}_j)$=$\mathbf{T}$. By reflexivity, $\mathbf{v}(\Diamond\Box\alpha,\mathbf{w}_j)$=$\mathbf{T}$. Then $\mathbf{v}(\Box\Diamond\Box\alpha,\mathbf{w}_i)$=$\mathbf{T}$. Finallly, since by the original supposition, $\mathbf{Rww}_i$, $\mathbf{v}(\Diamond\Box\Diamond\Box\alpha,\mathbf{w})$=$\mathbf{T}$.

For the converse, we prove the contrapositive. Suppose further that $\mathbf{v}(\Diamond\Box\alpha,\mathbf{w}$=$\mathbf{F}$. Then for every world $\mathbf{w}_i$ accessible to $\mathbf{w}$, $\mathbf{v}(\Box\alpha,\mathbf{w}_i)$=$\mathbf{F}$. And for every world $\mathbf{w}_j$ accessible to $\mathbf{w}_i$, $\mathbf{v}(\alpha,\mathbf{w}_j)$=$\mathbf{F}$. Now let $\mathbf{Rw}_j\mathbf{w}_k$. By transitivity, $\mathbf{v}(\alpha,\mathbf{w}_k)$=$\mathbf{F}$. Further, $\mathbf{v}(\Box\alpha,\mathbf{w}_k)$=$\mathbf{F}$, since transitivity requires that α have the value $\mathbf{F}$ at any world accessible to it. So $\mathbf{v}(\Diamond\Box\alpha,\mathbf{w}_j$=$\mathbf{F}$. Now since the choice of $\mathbf{w}_j$ is arbitrary, $\mathbf{v}(\Box\Diamond\Box\alpha,\mathbf{w}_i))=\mathbf{F}$, in which case $\mathbf{v}(\Diamond\Box\Diamond\Box\alpha,\mathbf{w}=\mathbf{F}$.

Suppose $\mathbf{v}(\Box\Diamond\Box\Diamond\alpha,\mathbf{w})=\mathbf{T}$ and that $\mathbf{Rww}_i$. Then $\mathbf{v}(\Diamond\Box\Diamond\alpha,\mathbf{w}_i)=\mathbf{T}$. Let $\mathbf{Rw}_i\mathbf{w}_j$, in which case $\mathbf{Rww}_j$. $\mathbf{v}(\Box\Diamond\alpha,\mathbf{w}_j=\mathbf{T}$. (That there is such a world is guaranteed by the seriality of $\mathbf{R}$.) So $\mathbf{v}(\Box\Box\Diamond\alpha,\mathbf{w}_i)=\mathbf{T}$. Because of reflexivity, applied twice, $\mathbf{v}(\Diamond\alpha,\mathbf{w}_i)=\mathbf{T}$. Since the choice of $\mathbf{w}_i$ is arbitrary, $\mathbf{v}(\Box\Diamond\alpha,\mathbf{w})=\mathbf{T}$.

For the converse, suppose that $\mathbf{v}(\Box\Diamond\alpha,\mathbf{w})=\mathbf{T}$ and that $\mathbf{Rww}_i$ and $\mathbf{Rw}_i\mathbf{w}_j$. By transitivity, $\mathbf{Rww}_j$. So $\mathbf{v}(\Diamond\alpha,\mathbf{w}_j)=\mathbf{T}$. Hence $\mathbf{v}(\Box\Diamond\alpha,\mathbf{w}_i)=\mathbf{T}$. By reflexivity, $\mathbf{v}(\Diamond\Box\Diamond\alpha,\mathbf{w}_i)=\mathbf{T}$. Finally, $\mathbf{v}(\Box\Diamond\Box\Diamond\alpha,\mathbf{w})=\mathbf{T}$ as the choice of $\mathbf{w}_i$ was arbitrary.

Exercise. Show how, with any version of Duality as a replacement rule, the impossibility version of SR(4) yields the same results as the necessity version.

1	$\Box\alpha$	Assumption
2	$\sim\Diamond\sim\alpha$	1 Duality
3	$\sim\sim\alpha$	1 SR(S4)
4	α	3 DN
5	$\Box\alpha$	3-4 $\Box$ I
6	$\Box\Box\alpha$	3-5 $\Box$ I

Exercise. Use derivations to establish the equivalences shown in the penultimate exercise.

1	$\Diamond\Box\alpha$	Assumption
2	$\Box\alpha$	Modal Assp
3	α	2 SR(S4)
4	$\Box\alpha$	3 $\Box$ I
5	$\Diamond\Box\alpha$	4 S $\Diamond$ I
6	$\Box\Diamond\Box\alpha$	3-5 $\Box$ I
7	$\Diamond\Box\Diamond\Box\alpha$	1 2-6 $\Diamond$ I

1	$\Box\Diamond\alpha$	Assp
2	$\Diamond\alpha$	1 SR(S4)
3	$\Box\Diamond\alpha$	3 $\Box$ I
4	$\Diamond\Box\Diamond\alpha$	3 S $\Diamond$ I
5	$\Box\Diamond\Box\Diamond\alpha$	2-4 $\Box$ I

1	$\Box\Diamond\Box\Diamond\alpha$	Assp
2	$\sim\Box\Diamond\alpha$	Assumption
3	$\Diamond\Box\sim\alpha$	2 Duality x2
4	$\Box\sim\alpha$	Modal Assp
5	$\sim\alpha$	4 SR(S4)
6	$\Box\sim\alpha$	5 $\Box$ I
7	$\Diamond\Box\sim\alpha$	6 S $\Diamond$ I
8	$\Box\Diamond\Box\sim\alpha$	5-7 $\Box$ I
9	$\Diamond\Box\Diamond\Box\sim\alpha$	3 4-8 $\Diamond$ I
10	$\sim\Box\Diamond\Box\Diamond\alpha$	9 Duality x 4
11	$\Box\Diamond\Box\Diamond\alpha$	1 R
12	$\Box\Diamond\alpha$	3-11 $\sim$ E

1	$\Diamond\Box\Diamond\Box\alpha$	Assp
2	$\sim\Diamond\Box\alpha$	Assp
3	$\Box\Diamond\sim\alpha$	2 Duality x2
4	$\Diamond\sim\alpha$	3 SR(S4)
5	$\Box\Diamond\sim\alpha$	4 $\Box$ I
6	$\Diamond\Box\Diamond\sim\alpha$	5 S $\Diamond$ I
7	$\Box\Diamond\Box\Diamond\sim\alpha$	4-6 $\Box$ I
8	$\sim\Diamond\Box\Diamond\Box\alpha$	7 Duality x4
9	$\Diamond\Box\Diamond\Box\alpha$	1 R
10	$\Diamond\Box\alpha$	3-9 $\sim$ E

Exercise. Show that SR(B)* is a derived rule given SR(B) and Duality as a replacement rule.

$\sim\alpha$	
$\Diamond\sim\alpha$	SR(B)
$\sim\Box\alpha$	Duality

Exercise. Suppose the operator 'F' is taken to represent what is the case now or in the future, and that 'G' indicates what is now and always will be. Would *B* generate results for Fα and Gα that would reasonably represent these notions?

A possible world would represent what is the case at a given time. We may suppose that this time is now. Then in this reading of *B*, what is now the case that α entails that it is now and always will be the case that it is now or will sometime be the case that α. It seems that what is now the case might not be the case in the future. So suppose that α is now the case but will not be the case after now. Then it will not always be the case that it will "now" (i.e., at that future time) or sometime thereafter be the case that α. So *B* remains unsuitable as a logic of time.

Exercise Prove that the modalities, $\Diamond\Box\Diamond\alpha$ and $\Box\Diamond\alpha$, are reducible to $\Diamond\alpha$, and the modalities $\Box\Diamond\Box\alpha$ and $\Diamond\Box\alpha$, are reducible to the $\Box\alpha$.

That $\Box\Diamond\alpha$ entails $\Diamond\alpha$ is trivial given that **R** is reflexive in *S5* semantics. Now suppose that $\mathbf{v}(\Diamond\alpha,\mathbf{w})=\mathbf{T}$. So for some accessible world, call it $\mathbf{w}_1$, $\mathbf{v}(\alpha,\mathbf{w}_1)=\mathbf{T}$. Suppose further that $\mathbf{Rww}_2$. Since **R** is Euclidean, $\mathbf{Rw}_2\mathbf{w}_1$, so $\mathbf{v}(\Diamond\alpha,\mathbf{w}_2)=\mathbf{T}$. Since the choice of $\mathbf{w}_2$ was arbitrary, $\mathbf{v}(\Box\Diamond\alpha,\mathbf{w}_2)=\mathbf{T}$.

Suppose $\mathbf{v}(\Diamond\Box\Diamond\alpha,\mathbf{w})=\mathbf{T}$. Then there is a world $\mathbf{w}_i$ accessible to **w** such that $\mathbf{v}(\Box\Diamond\alpha,\mathbf{w})=\mathbf{T}$. Since **R** is symmetrical, $\mathbf{v}(\Diamond\alpha,\mathbf{w})=\mathbf{T}$. For the converse, suppose $\mathbf{v}(\Diamond\alpha,\mathbf{w})=\mathbf{T}$. Then there is an accessible world $\mathbf{w}_1$ such that $\mathbf{v}(\alpha,\mathbf{w}_1)=\mathbf{T}$. Now suppose that $\mathbf{Rww}_2$, in which case $\mathbf{Rw}_2\mathbf{w}_1$. Then $\mathbf{v}(\Diamond\alpha,\mathbf{w}_2)=\mathbf{T}$. As the choice of $\mathbf{w}_2$ was arbitrary, $\mathbf{v}(\Box\Diamond\alpha, \mathbf{w})=\mathbf{T}$. Because **R** is reflexive, $\mathbf{v}(\Diamond\Box\Diamond\alpha, \mathbf{w})=\mathbf{T}$.

Suppose $\mathbf{v}(\Box\alpha,\mathbf{w})=\mathbf{T}$. By reflexivity, $\mathbf{v}(\Diamond\Box\alpha,\mathbf{w})=\mathbf{T}$. Conversely, suppose that $\mathbf{v}(\Diamond\Box\alpha,\mathbf{w})=\mathbf{T}$. Then there is an accessible world $\mathbf{w}_1$ such that $\mathbf{v}(\Box\alpha,\mathbf{w}_1)=\mathbf{T}$. Take an arbitrary world $\mathbf{w}_2$ accessible to $\mathbf{w}_1$, which, by the euclidean character of **R**, is accessible to $\mathbf{w}_1$. Then $\mathbf{v}(\alpha,\mathbf{w}_2)=\mathbf{T}$. So $\mathbf{v}(\Box\alpha,\mathbf{w})=\mathbf{T}$.

Suppose $\mathbf{v}(\Box\Diamond\Box\alpha,\mathbf{w})=\mathbf{T}$. Then because of reflexivity, $\mathbf{v}(\Diamond\Box\alpha,\mathbf{w})=\mathbf{T}$. It has already been shown that under this condition, $\mathbf{v}(\Box\alpha,\mathbf{w})=\mathbf{T}$. Conversely, suppose that $\mathbf{v}(\Box\alpha,\mathbf{w})=\mathbf{T}$. Now suppose $\mathbf{Rww}_1$. By symmetry, $\mathbf{v}(\Diamond\Box\alpha,\mathbf{w}_1)=\mathbf{T}$. Since the choice of $\mathbf{w}_1$ is arbitrary, $\mathbf{v}(\Box\Diamond\Box\alpha,\mathbf{w})=\mathbf{T}$.

Exercise. Use derivations to establish the equivalences just proved in the relative semantics for *S5*.

1	$\Box\Diamond\alpha$	Assumption
2	$\Diamond\alpha$	1 $\Box$ E

1	$\Diamond\alpha$	Assumption
2	$\|\ \Diamond\alpha$	1 SR(5)
3	$\Box\Diamond\alpha$	2 $\Box$ I

1	$\Diamond\alpha$	Assumption
2	$\|\ \Diamond\alpha$	1 SR(5)
3	$\Box\Diamond\alpha$	2 $\Box$ I
4	$\Diamond\Box\Diamond\alpha$	3 S $\Diamond$ I

1	$\Diamond\Box\Diamond\alpha$	Assumption
2	$\mid \sim\Diamond\alpha$	Assumption
3	$\mid \|\ \Diamond\sim\Diamond\alpha$	2 SR(B)
4	$\mid \Box\Diamond\sim\Diamond\alpha$	3 $\Box$ I
5	$\mid \sim\Diamond\Box\Diamond\alpha$	4 Duality x 2
6	$\mid \Diamond\Box\Diamond\alpha$	1 R
7	$\Diamond\alpha$	1-6 $\sim$ E

1	$\Box\alpha$	Assumption
2	$\Diamond\Box\alpha$	1 S $\Diamond$ I

1	$\Diamond\Box\alpha$	Assumption
2	$\mid \sim\Box\alpha$	Assumption
3	$\mid \Diamond\sim\alpha$	2 Duality
4	$\mid \|\ \Diamond\sim\alpha$	3 SR(5)
5	$\mid \Box\Diamond\sim\alpha$	4 $\Box$ I
6	$\mid \sim\Diamond\Box\alpha$	5 Duality x 2
6	$\Box\alpha$	2-6 $\sim$ E

1	$\Box\alpha$	Assumption
2	$\Diamond\Box\alpha$	1 S $\Diamond$ I
3	$\|\ \Diamond\Box\alpha$	2 SR(5)
4	$\Box\Diamond\Box\alpha$	3 $\Box$ I

1	$\vert\;\vert\; \Box\Diamond\Box\alpha$	Assumption
2	$\vert\;\vert\; \Diamond\Box\alpha$	1 $\Box$ E
3	$\vert\;\vert\;\vert\; \sim\Box\alpha$	Assumption
4	$\vert\;\vert\;\vert\; \Diamond\sim\alpha$	3 Duality
5	$\vert\;\vert\;\vert\; \Vert\; \Diamond\sim\alpha$	4 SR(5)
6	$\vert\;\vert\;\vert\; \Box\Diamond\sim\alpha$	5 $\Box$ I
7	$\vert\;\vert\;\vert\; \sim\Diamond\Box\alpha$	6 Duality x 2
8	$\vert\;\vert\;\vert\; \Diamond\Box\alpha$	1 R
9	$\vert\;\vert\; \Box\alpha$	2-8 $\sim$ E

Exercise. Show how the derivational system yields the characteristic derivation-relation for *S4.2*.

1	$\vert\;\vert\; \Diamond\Box\alpha$	Assumption
2	$\vert\;\vert\;\Vert\; \Diamond\Box\alpha$	1 SR(4.2)
3	$\vert\;\vert\;\Vert\;\Vert\; \Box\alpha$	Modal Assumption
4	$\vert\;\vert\;\Vert\;\Vert\; \alpha$	3 $\Box$ E
5	$\vert\;\vert\;\Vert\; \Diamond\alpha$	3-4 $\Diamond$ E
6	$\vert\;\vert\; \Box\Diamond\alpha$	2-5 $\Box$ I

Exercise. Use a semantical argument or a derivation (along with Lenzen's definition of belief) to show that $\{B\alpha\} \vDash_{S4.2} KB\alpha$ and $\{B_{s,t}\alpha\} \vDash_{S4.2} P_{s,t}K_{s,t}\alpha$.

By the definition of belief, we have the following (omitting subscripts for clarity). $\{\sim K\sim K\alpha\} \vDash_{S4.2} K\sim K\sim K\alpha$, and $\{\$ \sim K\sim K\alpha\} \vDash_{S4.2} \sim\sim K\sim K\sim \alpha$ (understainding $P\alpha$ as $\sim B\sim\alpha$). We can get a further reduction by using $F\alpha$ for $\sim K\sim K\alpha$ and Double Negation. $\{FK\alpha\} \vDash_{S4.2} KFK\alpha$, and $\{FK\alpha\} \vDash_{S4.2} KF\alpha$. We will show these relations using derivations in *S4.2*.

1	$\vert\; FK\alpha$	Assumption
2	$\vert\;\Vert\; FK\alpha$	1 SR(5)
3	$\vert\; KFK\alpha$	2 $\Box$ I

1	$\vert\;\vert\; FK\alpha$	Assumption
2	$\vert\;\vert\;\Vert\; FK\ \alpha$	1 SR(4.2)
3	$\vert\;\vert\;\Vert\;\Vert\; K\alpha$	Modal Assumption
4	$\vert\;\vert\;\Vert\;\Vert\; \alpha$	3 $\Box$ E
5	$\vert\;\vert\;\Vert\; F\alpha$	3-4 $\Diamond$ E
6	$\vert\;\vert\; KF\alpha$	2-5 $\Box$ I

Modal Predicate Logic

Exercise. Determine whether $\mathbf{d}_1$, $\mathbf{d}_3$, $\mathbf{d}_4$, satisfy '$(\forall x)(\exists y)Fxy$' on an interpretation where $\mathbf{D}=\{\mathbf{u}_1,\mathbf{u}_2\}$ and $\mathbf{v}(F,\mathbf{w})=\{\langle\mathbf{u}_1,\mathbf{u}_1\rangle, \langle\mathbf{u}_2,\mathbf{u}_2\rangle\}$.

$\mathbf{d}_1(x)=\mathbf{u}_1$ and $\mathbf{d}_1(y)=\mathbf{u}_1$,
$\mathbf{d}_3(x)=\mathbf{u}_2$ and $\mathbf{d}_3(y)=\mathbf{u}_1$,
$\mathbf{d}_4(x)=\mathbf{u}_2$ and $\mathbf{d}_4(y)=\mathbf{u}_2$.

All three variable-assignments satisfy the sentence, and all for the same reason. One type of x-variant of each of the three is $\mathbf{d}_i[\mathbf{u}_1/x]$. The other type of x-variant of each of the three is $\mathbf{d}_i[\mathbf{u}_2/\mathrm{x}]$. All instances of each of these types of x-variants satisify '(∃x)Fxy'. For $\mathbf{d}_i[\mathbf{u}_1/\mathrm{x}]$, one type of y-variant will be of the form $\mathbf{d}_i[\mathbf{u}_1/\mathrm{x}, \mathbf{u}_1/\mathrm{y}]$, which satisifies 'Fxy'. For $\mathbf{d}_i[\mathbf{u}_2/\mathrm{x}]$, we have $\mathbf{d}_i[\mathbf{u}_2/\mathrm{x}, \mathbf{u}_2/\mathrm{y}]$, which also satisfies 'Fxy'. Therefore in both cases, there is at least one y-variant which satisifies 'Fxy', and so in both cases, '(∃x)Fxy' is satisifed. In that case $\mathbf{d}_i$ satisifies '(∀x)(∃y)Fxy'.

Exercise. Using Duality and Quantifier Negation (from the derivational system for *PLI*), prove that any instance of $(\forall\mathbf{x})\Box\alpha(\mathbf{x}) \supset \Box(\forall\mathbf{x})\alpha(\mathbf{x})$ is an instance $\Diamond(\exists\mathbf{x})\alpha(\mathbf{x}) \supset (\exists\mathbf{x})\Diamond\alpha(\mathbf{x})$, and that any instance of $\Box(\forall\mathbf{x})\alpha(\mathbf{x}) \supset (\forall\mathbf{x})\Box\alpha(\mathbf{x})$ is an instance of $(\exists\mathbf{x})\Diamond\alpha(\mathbf{x}) \supset \Diamond(\exists\mathbf{x})\alpha(\mathbf{x})$.

$(\forall\mathbf{x})\Box\alpha(\mathbf{x}) \supset \Box(\forall\mathbf{x})\alpha(\mathbf{x})$ iff
$\sim\Box(\forall\mathbf{x})\alpha(\mathbf{x}) \supset \sim(\forall\mathbf{x})\Box\alpha(\mathbf{x})$ iff
$\Diamond\sim(\forall\mathbf{x})\alpha(\mathbf{x}) \supset \sim(\forall\mathbf{x})\Box\alpha(\mathbf{x})$ iff
$\Diamond(\exists\mathbf{x})\sim\alpha(\mathbf{x}) \supset \sim(\forall\mathbf{x})\Box\alpha(\mathbf{x})$ iff
$\Diamond(\exists\mathbf{x})\sim\alpha(\mathbf{x}) \supset (\exists\mathbf{x})\sim\Box\alpha(\mathbf{x})$ iff
$\Diamond(\exists\mathbf{x})\sim\alpha(\mathbf{x}) \supset (\exists\mathbf{x})\Diamond\sim\alpha(\mathbf{x})$

$\Box(\forall\mathbf{x})\alpha(\mathbf{x}) \supset (\forall\mathbf{x})\Box\alpha(\mathbf{x})$ iff
$\sim(\forall\mathbf{x})\Box\alpha(\mathbf{x}) \supset \sim\Box(\forall\mathbf{x})\alpha(\mathbf{x})$ iff
$(\exists\mathbf{x})\sim\Box\alpha(\mathbf{x}) \supset \sim\Box(\forall\mathbf{x})\alpha(\mathbf{x})$ iff
$(\exists\mathbf{x})\Diamond\sim\alpha(\mathbf{x}) \supset \sim\Box(\forall\mathbf{x})\alpha(\mathbf{x})$ iff
$(\exists\mathbf{x})\Diamond\sim\alpha(\mathbf{x}) \supset \Diamond\sim(\forall\mathbf{x})\alpha(\mathbf{x})$ iff
$(\exists\mathbf{x})\Diamond\sim\alpha(\mathbf{x}) \supset \Diamond(\exists\mathbf{x})\sim\alpha(\mathbf{x})$

Exercise. Prove this instance of the converse Barcan consequence: {□(∀x)Fx} $\vDash_{Q1-x}$ (∀x)□Fx'.

1	$\mathbf{v_d}\Box(\forall x)Fx, \mathbf{w})=\mathbf{T}$	Assp
2	$(\forall\mathbf{w}_i)(\mathbf{Rww}_i \supset \mathbf{v_d}(\forall x)Fx, \mathbf{w}_i)=\mathbf{T})$	1 ∀ I
3	$\mathbf{Rww}_1 \supset \mathbf{v_d}(\forall x)Fx, \mathbf{w}_1)=\mathbf{T}$	2 ∀ E
4	$\mathbf{u}_1 \in \mathbf{D}$	Assumption
5	$\mathbf{Rww}_1$	Assumption
6	$\mathbf{v_d}(\forall x)Fx, \mathbf{w}_1)=\mathbf{T}$	3 5 ⊃ E
7	$(\forall\mathbf{u}_i)(\mathbf{u}_i \in \mathbf{D} \supset \mathbf{v}_{\mathbf{d}[\mathbf{u}_i/x]}(Fx, \mathbf{w}_1)=\mathbf{T})$	5 ∀ TD (*Q1*)
8	$\mathbf{u}_1 \in \mathbf{D} \supset \mathbf{v}_{\mathbf{d}[\mathbf{u_1}/x]}(Fx, \mathbf{w}_1)=\mathbf{T}$	7 ∀E
9	$\mathbf{v}_{\mathbf{d}[\mathbf{u_1}/x]}(Fx, \mathbf{w}_1)=\mathbf{T}$	4 8 ⊃ E
10	$\mathbf{Rww}_1 \supset \mathbf{v}_{\mathbf{d}[\mathbf{u_1}/x]}(Fx, \mathbf{w}_1)=\mathbf{T}$	5-9 ⊃I
11	$(\forall\mathbf{w}_i)(\mathbf{Rww}_i \supset \mathbf{v}_{\mathbf{d}[\mathbf{u_1}/x]}(Fx, \mathbf{w}_i)=\mathbf{T})$	10 ∀I
12	$\mathbf{v}_{\mathbf{d}[\mathbf{u_1}/x]}(\Box Fx, \mathbf{w})=\mathbf{T}$	11 □ TD
13	$\mathbf{u}_1 \in \mathbf{D} \supset \mathbf{v}_{\mathbf{d}[\mathbf{u_1}/x]}(\Box Fx, \mathbf{w})=\mathbf{T}$	4-12 ⊃ I
14	$(\forall\mathbf{u}_i)(\mathbf{u}_i \in \mathbf{D} \supset \mathbf{v}_{\mathbf{d}[\mathbf{u_i}/x]}(\Box Fx, \mathbf{w})=\mathbf{T})$	13 ∀ I
15	$\mathbf{v_d}(\forall x)\Box Fx, \mathbf{w})=\mathbf{T}$	14 ∀TD (*Q1*)

Exercise. Show that {(∃x)□Fx} $\vDash_{Q1-x}$ □(∃x)Fx.

Suppose $\mathbf{v_d}((\exists x)\Box Fx,\mathbf{w})=\mathbf{T}$. Then for some $\mathbf{d}[\mathbf{u}_i/x]$ which is an x-variant of

$\mathbf{d}$, $\mathbf{v}_{\mathbf{d}[\mathbf{u}_i/\mathrm{x}]}$(□Fx,$\mathbf{w}$)=$\mathbf{T}$. So for all worlds $\mathbf{w}_i$ accessible to $\mathbf{w}$, $\mathbf{v}_{\mathbf{d}[\mathbf{u}_i/\mathrm{x}]}$(Fx,$\mathbf{w}$)=$\mathbf{T}$. Hence, there is an x-variant of $\mathbf{d}$ which satisifies 'Fx', and so $\mathbf{v}_{\mathbf{d}}$((∃x)Fx,$\mathbf{w}_i$)=$\mathbf{T}$. Since the choice of $\mathbf{w}_i$ is arbitrary, this holds at all accessible worlds, and so $\mathbf{v}_{\mathbf{d}}$(□(∃x)Fx,$\mathbf{w}$)=$\mathbf{T}$.

Exercise. Show that $\vDash_{Q1-x}$ (∀x)□x=x directly using the *Q1* semantics.

The assignment made by $\mathbf{d}$ to 'x' is the same as itself: $\mathbf{d}$(x)=$\mathbf{d}$(x). So $\mathbf{d}$(x,$\mathbf{w}$)=$\mathbf{d}$(x,$\mathbf{w}$), for any world $\mathbf{w}$ in $\mathbf{W}$. Therefore, $\mathbf{v}_{\mathbf{d}}$(x,$\mathbf{w}$)=$\mathbf{v}_{\mathbf{d}}$(x,$\mathbf{w}$). In that case, $\mathbf{v}_{\mathbf{d}}$(x=x,$\mathbf{w}$)=$\mathbf{T}$. Since this holds at all worlds in $\mathbf{W}$, it holds at all worlds accessible to $\mathbf{w}$, so $\mathbf{v}_{\mathbf{d}}$(□x=x,$\mathbf{w}$)=$\mathbf{T}$. Moreover, as $\mathbf{d}$(x) is arbitrary, this reasoning holds for all x-variants of $\mathbf{d}$, in which case $\mathbf{v}_{\mathbf{d}}$((∀x)□x=x,$\mathbf{w}$)=$\mathbf{T}$.

Exercises. Give derivations to show that the following consequences hold. {(∃x)◊Fx}$\vdash_{Q1-x}$ ◊(∃x)Fx, {(∃x)□Fx}$\vdash_{Q1-x}$ □(∃x)Fx.

1	(∃x)◊Fx	Assumption
2	◊Fa	Assumption
3	Fa	Modal Assumption
4	(∃x)Fx	3 ∃ I
5	◊(∃x)Fx	3-4 ◊ E
6	◊(∃x)Fx	1 2-5 ∃ E

1	(∃x)□Fx	Assumption
2	□Fa	Assumption
3	Fa	2 SR
4	(∃x)Fx	3 ∃ I
5	□(∃x)Fx	3-4 □ I
6	□(∃x)Fx	1 2-5 ∃ E

Exercise. Show that the Barcan consequence holds if we assume that the accessibility relation is symmetrical and that the UD of a world is a subset of the UD of any accessible world.

Given that accessibility is symmetrical and that the UD of a world is the UD of any accessible world, the UDs of any given world and any accessible world are identical: each is a subset of the other. In that case, the original reasoning for the Barcan consequence can be used. If everything at the UD in a world satisfies the condition '□Fx', then all those things satisfy 'Fx' at all accessible worlds. But the UD of any accessible world is the same as the UD at the given world, so '(∀x)Fx' is true at such a world, in which case '□(∀x)Fx' is true at the original world.

Exercise. Justify the soundness in *QPL* of the inference from '□Fa' to '(∃x)□Fx'.

Suppose that '□Fa' is true at $\mathbf{w}$. Then 'Fa' has a truth-value at $\mathbf{w}$, in which case 'a' designates an object in the UD of $\mathbf{w}$. Moreover, by the nesting condition, 'a' also designates an object in the UD of any accessible world $\mathbf{w}_i$. That same object will be designated by an x-variant of any variable-assignment $\mathbf{d}$ at each of the accessible worlds. So because 'Fa' is true at $\mathbf{w}_i$, so is 'Fx' on an x-variant of **d**. In that case '$\Box$Fx' is for an x-variants of **d** at **w**, so that '(∃x)$\Box$Fx' is true at **w**.

Exercise. Show that '$\Box$(∃x)x=a' is invalid in any *FQ1R* system.

Consider a world **w** to which $\mathbf{w}_i$ is the only accessible world. (World $\mathbf{w}_i$ can be **w** if **R** is reflexive.) Let 'a' not designate anything in the UD of $\mathbf{w}_i$. In that case, for no x-variant of a given variable-assignment **d** is it the case that 'x=a' is true at $\mathbf{w}_1$. So '(∃x)Fx' is false at $\mathbf{w}_i$, and hence '$\Box$(∃x)Fx' is false at **w**.

Exercise. Show the validity in all *FQ1R* systems of '(∀x)(∀y)(x=y ⊃ $\Box$x=y)'.

Suppose 'x=y' is true at a given world **w**. By the truth-definition for identity, 'x' and 'y' are assigned the same member of **D** (whether the object exists in the UD of that world or not). Now consider a $\mathbf{w}_i$ accessible to **w**. At $\mathbf{w}_i$, 'x=y' is true on any such variable assignment, so '$\Box$x=y' is true at **w**. Hence, if 'x=y' is true at **w**, then '$\Box$x=y' is true at **w**, in which case 'x=y ⊃ $\Box$x=y' is true at **w**. This holds no matter what object is assigned to 'x' and 'y', so '(∀x)(∀y)(x=y ⊃ $\Box$x=y)' is true at **w**.