Basic Probability Calculus

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Closure

1a. If p and q are assigned probabilities, then so are all of their logical (Boolean) combinations formed with 'not,' 'and' and 'or':

not-p, p and q, p or q

1b. If p and q are assigned probabilities, then so is the conditional probability of q given p:

$P(q/p)$

Limits

2a. $P(p) = 0$ iff p cannot be true
2b. $P(p) = 1$ iff p cannot be false
2c. $0 \leq P(p) \leq 1$

Example: P(heads on this toss and tails on this toss) = 0

Negation

3. $P(\text{not-}p) = 1 - P(p)$

Example: Suppose P(heads on this toss) = .5. Then P(not heads on this toss) = 1 - .5 = .5. Given that tails come up on a toss if and only if heads do not come up, then P(tails on this toss) = .5

Simple Disjunction

4. If $P(p \text{ and } q) = 0$, then $P(p \text{ or } q) = P(p) + P(q)$

Example: P(heads on this toss and tails on this toss) = 0. P(heads on this toss or tails on this toss) = .5 + .5 = 1

Independence

$P(p)$ is independent of $P(q)$ if and only if $P(p/q) = P(p)$

Example: P(heads on the second toss) = P(heads on the second toss / heads on the first toss)

Conjunction

5. $P(p \text{ and } q) = P(p) \times P(q/p)$

Example: P(heads on the first toss and heads on the second toss) = P(heads on the first toss) × P(heads on the second toss / heads on the first toss) = P(heads on the first toss) × P(heads on the second toss) = .5 × .5 = .25

Example: Suppose that the probability of any given side coming up on any given throw of a six-sided die is 1/6. Then P(this throw comes up a six) = 1/6. Further, given the probability of a die coming up two numbers at once is 0 and that the outcomes of throws are independent, P(this throw comes up even) = P(this throw comes up a two) + P(this throw comes up a four) + P(this throw comes up a six) = 1/6 + 1/6 + 1/6 = 1/2.

P(this throw comes up even / this throw comes up a six) = 1. So, P(this throw comes up a six and this throw comes up even) = P(this throw comes up a six) × P(this throw comes up even / this throw comes up a six) = 1/6 × 1 = 1/6.

Alteratively, P(this throw comes up a six / this throw comes up even) = 1/3. So P(this throw comes up even and this throw comes up a six) = P(this throw comes up even) × P(this throw comes up a six / this throw comes up even) = 1/2 × 1/3 = 1/6.

The Lottery Paradox

Suppose there is a "fair" lottery, in which one ticket has been drawn from a pool of 1,000. We will suppose, then that for each ticket, the probability that that ticket has been drawn is one in one thousand. Let 'T_1' indicate that ticket number 1 has been drawn, and so on for each of the other tickets. Then $P(T_1) = .001$, and the same holds through T_{1000}. If follows from the probability calculus that $P(\text{not-}T_1)$=.999, and the same holds through T_{1000}.

Now suppose that a probability of .999 is sufficient for justification. In that case, the belief that not-T_1 is a justified belief. And so is the belief that not-T_2, etc. If we accept a further principle, that if one is justified in believing all the conjuncts of a conjunction, one is justified in believing the conjunction itself, there is a problem. Since not all the tickets are losers, $P(\text{not-}T_1$ and not-T_2 and $\ldots$ and not-$T_{1000}) = 0$, and so one is not justified in believing the conjunction despite being justified in believing all the conjuncts. This is the lottery paradox. Note that it does not depend on the specific numbers used here. It works no matter how many tickets are entered in the lottery, and consequently no matter how high $P(T_1)$ is.

Various suggestions have been made to evade the paradox. One could, for example, throw out the principle that justified belief in the conjucts of a conjuction implies a justified belief in the conjunction. Lehrer solves the paradox by denying one of its suppositions, i.e., that high probability is sufficient for justification. In the lottery case, the various results of the draw are not independent. The assumption that one of the tickets is a loser diminishes the probability that another is. For example, $P(\text{not-}T_1 \ / \ \text{not-}T_2)$ = 998/999, which is less than 999/1000. The information that not-T_2 is *negatively relevant* to the information that not-T_1.

The problem for justification, as Lehrer sees it, is that there is no way to rule out the negatively relevant information merely on the basis of probabilities. Specifically, not-T_1 is no more probable than the negatively relevant not-T2. The idea that a justified belief must be more plausible (in this case, more probable) than all negatively relevant information is the core of Lehrer's theory of justification.